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ON DOUBLE POLYADICS, WITH APPLICATION TO THE LINEAR MATRIX EQUATION.

By Frank L. Hitchcock.

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1. Gibbs' Concept of Dyadics and Polyadics.

A system of Vector Algebra in N dimensions following Gibbs is based on a set of N unit vectors $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N$ such that the dot products $\mathbf{e}_i \cdot \mathbf{e}_k$ are unity when subscripts are equal, otherwise zero. Any other vector $\mathbf{a}$ may be expressed in terms of the fundamental units,

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_N\mathbf{e}_N. \tag{1}$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ written together with no dot or other sign between them constitute a dyad $\mathbf{ab}$ called the indeterminate product of $\mathbf{a}$ into $\mathbf{b}$. A dyad $\mathbf{e}_i\mathbf{e}_k$ will be called a fundamental dyad or dyad unit. If we have any dyad $\mathbf{ab}$ the first vector $\mathbf{a}$ is called the antecedent, the second the consequent. Both $\mathbf{a}$ and $\mathbf{b}$ will be called factors of the dyad. A dyad is zero when and only when all products of scalar elements a_ib_k are zero, that is, when one of the factors vanishes in all its elements.

A dyadic is a sum of dyads, or, with no increase in generality, a sum of dyads each multiplied by a scalar factor. The *elements* of a dyadic are the sums $\Sigma a_i b_k$ of corresponding products taken from each dyad term, where, for an element A_{ik} of a dyadic $\mathbf{A}$ the subscripts i and k are constant and the summation is over the various dyad terms. Thus if

$$\mathbf{A} = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots + \mathbf{a}_h\mathbf{b}_h \tag{2}$$

we have

$$A_{ik} = a_{1i}b_{1k} + a_{2i}b_{2k} + \cdots + a_{hi}b_{hk}. \tag{3}$$

It is frequently of use to write the dyad $\mathbf{ab}$ as symbolical of the dyadic and $a_i b_k$ as symbolical of the element A_{ik}, omitting the subscript h and the sign of summation with respect to h.[1]

A dyadic may be written in terms of the dyad units $\mathbf{e}_i\mathbf{e}_k$ *thus*

$$\mathbf{A} = \Sigma A_{ik}\mathbf{e}_i\mathbf{e}_k \tag{4}$$

where both subscripts run from 1 to N.

In a similar manner several vectors $\mathbf{abc}\cdots\mathbf{g}$ written with no sign between them constitute a polyad. When we wish to indicate that the polyad is of order K, that is, it is the indeterminate product of K vectors, we may call it a K-ad. The vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc., in order, will be called the first, second, third, etc. factors of the polyad. A polyadic is a sum of polyads, all of the same order. The polyads $\mathbf{e}_p\mathbf{e}_q\mathbf{e}_r\cdots\mathbf{e}_s$ to K factors will be called the fundamental polyads of order K or fundamental K-ads. It is evident that any polyadic may be expanded in terms of the fundamental polyads of its own order. The elements of a polyadic are the scalar coefficients in this expansion. Thus if we have K subscripts $p, q, \cdots, s$, all of which run from 1 to N, we may write a K-adic $\mathbf{A}$ as

$$\mathbf{A} = \Sigma A_{pq\cdots s}\mathbf{e}_p\mathbf{e}_q\cdots\mathbf{e}_s. \tag{5}$$

It is thus apparent that in general a K-adic depends on N^K scalar elements. These may be primarily regarded as forming a K-dimensional block, and most conveniently written by means of adjacent squares when special values have to be assigned. For example if $N = 2$ and $K = 4$ we should have the scheme of elements

A_{1111}, A_{1112} A_{1121}, A_{1122}	A_{1211}, A_{1212} A_{1221}, A_{1222}
A_{2111}, A_{2112} A_{2121}, A_{2122}	A_{2211}, A_{2212} A_{2221}, A_{2222}

as one possible way of representing a four-dimensional assemblage of elements on a flat surface.

A polyadic is defined to be zero when and only when all its elements are zero. Since our fundamental conception of a polyadic is as a sum of polyads rather than as a mere aggregation of scalars, the equation $\mathbf{A} = 0$ is to be thought of as equivalent to N^K equations of the form

$$a_{1p}b_{1q}\cdots g_{1s} + a_{2p}b_{2q}\cdots g_{2s} + \cdots + a_{hp}b_{hq}\cdots g_{hs} = 0\,. \qquad (7)$$

It is frequently useful to indicate such a set of equations symbolically as $\mathbf{ab}\cdots\mathbf{g} = 0$, where, as before, the polyad is written symbolically for the polyadic.

2. Polyadics as Vectors in Space of Higher Dimensions.

Investigations on polyadics may be distinguished according to whether the K-dimensional character is important or not. If we agree on some definite order among the fundamental polyads we may write

$$\mathbf{E}_r = \mathbf{e}_p\mathbf{e}_q\cdots\mathbf{e}_s \qquad (8)$$

and let r run from 1 to N^K while each of the K subscripts on the right runs from 1 to N. With $A_r = A_{pq\ldots s}$ we shall then have

$$\mathbf{A} = A_1\mathbf{E}_1 + A_2\mathbf{E}_2 + \cdots + A_n\mathbf{E}_n \qquad (9)$$

where, for convenience, n has been put for N^K.

The polyadic $\mathbf{A}$ thus takes the form of a vector in space of n dimensions. This concept is justified if we introduce the multiple dot product defined by

$$(\mathbf{e}_a\mathbf{e}_b\cdots\mathbf{e}_g) : (\mathbf{e}_p\mathbf{e}_q\cdots\mathbf{e}_s) = (\mathbf{e}_a\cdot\mathbf{e}_p)\,(\mathbf{e}_b\cdot\mathbf{e}_q)\cdots(\mathbf{e}_g\cdot\mathbf{e}_s) \qquad (10)$$

where the colon in every case indicates K-tuple dot product. In words

Definition. The K-tuple dot product of two K-ads is the product of dot products of corresponding factors.

The K-tuple dot product of two K-adics is the sum of K-tuple dot products term by term. By virtue of the distributive law of multiplication the relation (10) is symbolical of the K-tuple dot multiplication of two K-adics. When $K = 2$ we have Gibbs' double dot multiplication.

By (8) and (10) it is evident that $\mathbf{E}_i : \mathbf{E}_k$ is unity when $i = k$, otherwise zero. Thus the fundamental polyads behave with respect to multiple dot product as do unit vectors $\mathbf{e}_1, \mathbf{e}_2, \cdots \mathbf{e}_n$ in space of n

dimensions with respect to ordinary dot product. Therefore all properties of such vectors which pertain to dot multiplication alone will go over into properties of polyadics and multiple dot product.

To say the same thing in another way, the fundamental polyads $\mathbf{e}_p\mathbf{e}_q\cdots\mathbf{e}_s$ form a normal orthogonal system with respect to multiple dot multiplication, in the sense that the product of two unlike polyads of order K is zero while the product of a polyad by itself is unity. The notation $\mathbf{E}_r$ for the polyad is convenient but not essential. The analogy with $\mathbf{e}_r$, while of much value in a formal sense, is temporary in character, to be employed or laid aside according as we wish to forget or to emphasize the K-tuple nature of the polyads.

3. The Fundamental Identity of Dot Multiplication.

Let there now be a set of N^K or n polyadics of order K as $\mathbf{A}_1, \mathbf{A}_2, \cdots \mathbf{A}_n$, arbitrarily chosen. Any one of these, as $\mathbf{A}_i$, may be expanded in the manner of (9), thus

$$\mathbf{A}_i = a_{i1}\mathbf{E}_1 + a_{i2}\mathbf{E}_2 + \cdots + a_{in}\mathbf{E}_n. \tag{11}$$

Let there also be another polyadic which we may on occasion call $\mathbf{A}_{n+1}$ but which it will be more convenient at present to call $\mathbf{M}$ and to expand in the form

$$\mathbf{M} = M_1\mathbf{E}_1 + M_2\mathbf{E}_2 + \cdots + M_n\mathbf{E}_n. \tag{12}$$

If we now take the multiple dot product $\mathbf{M}:\mathbf{A}_i$, remembering the principle of orthogonality pointed out in the last article, we shall have

$$\mathbf{M}:\mathbf{A}_i = M_1a_{i1} + M_2a_{i2} + \cdots + M_na_{in}. \tag{13}$$

There are n equations of this form, one for each of the polyadics $\mathbf{A}_1\cdots\mathbf{A}_n$. Together with (12) we thus have $n+1$ equations linear in the scalars $M_1\cdots M_n$. It is true that (12) is a K-adic equation and is itself equivalent to n scalar equations. But since all the equations (13) are merely scalar equations, it is not hard to see that the determinant

$$\begin{vmatrix} \mathbf{M} & , \mathbf{E}_1, & \mathbf{E}_2, \cdots, & \mathbf{E}_n \\ \mathbf{M}:\mathbf{A}_1, & a_{11}, & a_{12}, \cdots, & a_{1n} \\ \mathbf{M}:\mathbf{A}_2, & a_{21}, & a_{22}, \cdots, & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{M}:\mathbf{A}_n, & a_{n1}, & a_{n2}, \cdots, & a_{nn} \end{vmatrix} \tag{14}$$

must vanish; because when polyadics have to be multiplied by scalars only, all the laws of ordinary algebra are obeyed.

We have now to develop this determinant by the elements of the first column, yielding the identity

$$C_0\mathbf{M} + \mathbf{C}_1\mathbf{M}:\mathbf{A}_1 + \mathbf{C}_2\mathbf{M}:\mathbf{A}_2 + \cdots + \mathbf{C}_n\mathbf{M}:\mathbf{A}_n = 0 \qquad (15)$$

where the scalar C_0 and the polyadics $\mathbf{C}_1 \cdots \mathbf{C}_n$ are the cofactors of the elements into which they are multiplied. Evidently C_0 is the determinant of the n^2 elements a_{ik}. To exhibit the character of $\mathbf{C}_1$, $\mathbf{C}_2$, etc., let c_{ik} denote the cofactor of a_{ik} from the n-rowed determinant C_0. On developing (14) it appears that

$$\mathbf{C}_i = -(c_{i1}\mathbf{E}_1 + c_{i2}\mathbf{E}_2 + \cdots + c_{in}\mathbf{E}_n) \qquad (16)$$

whence the $\mathbf{C}_i$ are polyadics whose elements are the *negatives* of the $n-1$ rowed cofactors from C_0.

So far the $n+1$ polyadics $\mathbf{M}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$ have been quite arbitrary. If, however, the $\mathbf{A}_i$ are linearly independent, their determinant C_0 is not zero and we may introduce the important new set of polyadics $\mathbf{A}'_1, \mathbf{A}'_2 \cdots, \mathbf{A}'_n$ defined by

$$C_0\mathbf{A}'_i = -\mathbf{C}_i \qquad (17)$$

or by (16)

$$\mathbf{A}'_i = \frac{1}{C_0}(c_{i1}\mathbf{E}_1 + c_{i2}\mathbf{E}_2 + \cdots + c_{in}\mathbf{E}_n). \qquad (18)$$

The set $\mathbf{A}'_i$ is said to be reciprocal to the set $\mathbf{A}_i$.

By multiplying the expansions (11) and (18), remembering that c_{ik} is the cofactor of a_{ik} from the determinant C_0, we have the relation $\mathbf{A}'_i:\mathbf{A}_k = 1$ when subscripts are equal, otherwise zero, which we may also express by saying that a set of n polyadics and its reciprocal set with respect to multiple dot product are bi-orthogonal.

The C_i may now be eliminated by the aid of (17); and the fundamental identity (15), (on the hypothesis that C_0 does not vanish), rearranged in the form

$$\mathbf{M} = (\mathbf{A}'_1\mathbf{A}_1 + \mathbf{A}'_2\mathbf{A}_2 + \cdots + \mathbf{A}'_n\mathbf{A}_n):\mathbf{M}. \qquad (19)$$

The expression in parentheses will be denoted by I; it is an idemfactor in the sense that its K-tuple dot product into an arbitrary K-adic leaves that K-adic unaltered; and it is the most important special case of a *double polyadic,— a dyadic whose antecedents and consequents are polyadics.*

Let the expression in parentheses be now transformed by putting in the values of all the polyadics $\mathbf{A}$ and $\mathbf{A}'$ from the expansions (11)

and (18). Collecting the scalar coefficient of $\mathbf{E}_k\mathbf{E}_k$ we find it to be $\frac{1}{C_0}\Sigma c_{ik}a_{ik}$ *summed on* i, which is unity. Again, the coefficient of $\mathbf{E}_j\mathbf{E}_k$ when j and k are unequal is $\frac{1}{C_0}\Sigma c_{ij}a_{ik}$ *summed on* i, which is zero. Hence $I = \Sigma\mathbf{E}_k\mathbf{E}_k$ *summed on* k, and this latter expression for the idemfactor is symmetrical in the sense that it is unaltered by interchanging antecedents and consequents. It follows that the reciprocal relation between two sets of polyadics is a mutual one, and we have

$$I = \Sigma\mathbf{E}_k\mathbf{E}_k = \Sigma\mathbf{A}'_k\mathbf{A}_k = \Sigma\mathbf{A}_k\mathbf{A}'_k \tag{20}$$

and also

$$\mathbf{M} = I:\mathbf{M} = \mathbf{M}:I \tag{21}$$

or the idemfactor may be used either as prefactor or as postfactor.

To illustrate, if we take $K = 1$ and $N = 3$, our polyadics reduce to vectors in ordinary space; and if our unit vectors be the usual **i**, **j**, and **k** the idemfactor becomes the usual idemfactor of Gibbs, namely **ii** + **jj** + **kk**. If $K = 2$ and N = 3 with the same notation, our polyadics are ordinary dyadics; the idemfactor (20) is a double dyadic, having nine terms obtained by doubling the nine fundamental dyads, viz.

$$I = \mathbf{iiii} + \mathbf{jjjj} + \mathbf{kkkk} + \mathbf{ijij} + \mathbf{jiji} + \mathbf{jkjk} + \mathbf{kjkj} + \mathbf{kiki} + \mathbf{ikik} \tag{22}$$

and it may easily be verified that the double dot product of any of the nine fundamental dyads **ii**, **ij**, etc., either *by* or *into* this expression gives the dyad unchanged; whence the same is true for any dyadic.

4. Double Polyadics.

Two polyadics **A** and **M** written together with no sign between them will constitute a double polyad **AM**. A sum of double polyads is a double polyadic, assuming always that the terms of the sum are of like character. In the present investigation it will always be assumed that the factors **A** and **M** of all double polyads are polyadics of the same order K. When desirable, all the antecedents **A** may be expanded in terms of the fundamental K-ads $\mathbf{E}_i$, and the consequents collected so that any double polyadic φ may be written in the form

$$\varphi = \mathbf{E}_1\mathbf{M}_1 + \mathbf{E}_2\mathbf{M}_2 + \cdots + \mathbf{E}_n\mathbf{M}_n. \tag{23}$$

Each of the consequents $\mathbf{M}_i$ may be expanded in terms of the $\mathbf{E}_i$ by aid of the notation

$$\mathbf{M}_i = m_{i1}\mathbf{E}_1 + m_{i2}\mathbf{E}_2 + \cdots + m_{in}\mathbf{E}_n. \tag{24}$$

If the $\mathbf{M}_i$ of (23) be expanded by (24) the complete development of φ will be in terms of the double polyad units $\mathbf{E}_i\mathbf{E}_k$ or

$$\varphi = \Sigma m_{ik}\mathbf{E}_i\mathbf{E}_k, \ [i, k, = 1, 2, \cdots n]. \tag{25}$$

It is clear that a double polyadic depends on n^2, that is N^{2K} scalar elements. Thus double dyadics in ordinary space, of which the idemfactor (22) is a special case, would yield in general 81 terms if it were necessary to expand completely, which fortunately it is not. A double dyadic is of course a tetradic. If need arose to emphasize the $2K$-adic character of a double polyadic and at the same time to depict the scalar elements, use could be made of a system of adjacent squares like (6). In the applications which follow, however, the scalar elements of a double polyadic are to be thought of as a binary assemblage or square array. For example if $K = 2$ and $N = 2$, with $\mathbf{i}$ and $\mathbf{j}$ for unit vectors, the order of the fundamental dyads may be agreed upon by taking $\mathbf{E}_1 = \mathbf{ii}$, $\mathbf{E}_2 = \mathbf{ij}$, $\mathbf{E}_3 = \mathbf{ji}$ and $\mathbf{E}_4 = \mathbf{jj}$. The corresponding scalars m_{pq} will then be arranged as

$$\begin{matrix} m_{11}, & m_{12}, & m_{13}, & m_{14} \\ m_{21}, & m_{22}, & m_{23}, & m_{24} \\ m_{31}, & m_{32}, & m_{33}, & m_{34} \\ m_{41}, & m_{42}, & m_{43}, & m_{44} \end{matrix} \tag{26}$$

where m_{pq} is the coefficient of $\mathbf{E}_p\mathbf{E}_q$ in the double dyadic $\Sigma m_{pq}\mathbf{E}_p\mathbf{E}_q$. Comparing with the four-subscript arrangement (6), we see that the upper left hand square of (6) would correspond to the first row of (26), the upper right hand square of (6) to the second row of (26), and so on.

Returning to our fundamental concept of a double polyadic as a sum of dyads whose antecedents and consequents are polyadics, we next define the K-tuple dot product $\varphi_1 : \varphi_2$ of two double polyadics, (in precise analogy with Gibbs' dot product of two dyadics), to be the sum of all terms of the form $\mathbf{AM} : \mathbf{BN}$ or $(\mathbf{M} : \mathbf{B})\ \mathbf{AN}$ obtained by multiplying out the two double polyadics and taking the K-tuple dot product of each consequent of φ_1 into each antecedent of φ_2. Thus $\varphi_1 : \varphi_2$ is also a double K-adic. We abbreviate $\varphi : \varphi$ as φ^2, and $\varphi : \varphi : \varphi$ as φ^3, etc. This multiplication is easily seen to be associative, as well as distributive, but of course not generally commutative.

5. The Hamilton-Cayley Equation for Double Polyadics.

Let φ be a double K-adic expanded as in (23) so that the antecedents are the fundamental K-ads $\mathbf{E}_1 \cdots \mathbf{E}_n$. Let Ψ be a second double K-adic $\Sigma\mathbf{BN}$ and let its consequents first be expanded in terms of $\mathbf{E}_1 \cdots \mathbf{E}_n$ and terms collected so that Ψ may take the form

$$\Psi = \mathbf{B}_1\mathbf{E}_1 + \mathbf{B}_2\mathbf{E}_2 + \cdots + \mathbf{B}_n\mathbf{E}_n \tag{27}$$

so that the consequents are the fundamental K-ads. If we now form the K-tuple dot product $\varphi:\Psi$ it is clear that the scalar coefficient of $\mathbf{E}_i\mathbf{E}_k$ will be $\mathbf{M}_i:\mathbf{B}_k$ or, in terms of the scalar elements of the polyadics, will be $\Sigma m_{is}b_{sk}$ summed on s. This agrees in form with the law of multiplication of matrices of order n, a result which might have anticipated; for if vectors are analogous to polyadics we should expect matrices to behave like double polyadics, just as matrices of order N behave like ordinary dyadics. The analogy holds of course only so long as we are concerned with formal laws possessed in common by the two algorisms.

By virtue of this analogy, however, it is evident without further proof that any double polyadic φ must satisfy an identity of the same form as the Hamilton-Cayley equation belonging to a square matrix of order n, namely

$$\varphi^n - m_1\varphi^{n-1} + m_2\varphi^{n-2} - \cdots + (-1)^n m_n I = 0 \tag{28}$$

where the coefficients $m_1 \cdots m_n$ are scalars. The determinant of the matrix or double K-adic $\varphi - gI$, where g is a scalar, is

$$\begin{vmatrix} m_{11} - g, & m_{12}, & m_{13}, & \cdots, & m_{1n} \\ m_{21}, & m_{22} - g, & m_{23}, & \cdots, & m_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ m_{n1}, & m_{n2}, & m_{n3}, & \cdots, & m_{nn} - g \end{vmatrix} \tag{29}$$

and by writing φ in place of g and equating to zero we have the Hamilton-Cayley equation.[2]

It is by no means necessary, however, to expand in terms of the unit polyads in order to form the Hamilton-Cayley equation. Consider for a moment the case $K = 1$, $N = 3$, when the double polyadic becomes Gibbs' dyadic in ordinary space. The coefficient m_1 in the Hamilton-Cayley equation becomes Gibbs' φ_S which, if the dyadic be $\Sigma\mathbf{ab}$ is $\Sigma\mathbf{a}\cdot\mathbf{b}$, or symbolically $m_1 = \mathbf{a}\cdot\mathbf{b}$. Again, the coefficient m_2

becomes half $(\varphi \overset{\times}{\times} \varphi)$ or φ_{2S} which, if $\mathbf{ab}$ and $\mathbf{cd}$ are any two terms of the dyadic, is the sum of all terms of the form $\mathbf{a}\cdot\mathbf{bc}\cdot\mathbf{d} - \mathbf{a}\cdot\mathbf{dc}\cdot\mathbf{b}$, or symbolically

$$2m_2 = \mathbf{a}\cdot\mathbf{bc}\cdot\mathbf{d} - \mathbf{a}\cdot\mathbf{dc}\cdot\mathbf{b}. \tag{30}$$

The case of dyadics whose antecedents and consequents are combinatorial products of vectors in N-space, the sum of the classes of antecedent and consequent being equal to the dimensionality of the space, has also been treated;[3] and by applying multiple algebra to a projective point space of three dimensions, various types of dyadics occur, with corresponding Hamilton-Cayley equations.[4]

By analogy it is plain we shall expect m_1 for the double polyadic to be the sum of $\mathbf{A} : \mathbf{M}$, the K-tuple dot product of each antecedent into its own consequent. Using the expansion (23) and noting that $\mathbf{E}_i : \mathbf{M}_i = m_{ii}$ we have $\Sigma\mathbf{A} : \mathbf{M} = \Sigma m_{ii}$ which is the same as m_1 by the usual matrix theory.[5] If we agree to write, by analogy with Gibbs, $\Sigma\mathbf{A} : \mathbf{M} = \varphi_S$, called the scalar of φ, we shall have

$$m_1 = \Sigma\mathbf{A} : \mathbf{M} = \varphi_S. \tag{31}$$

By analogy with Gibbs' $(\varphi \overset{\times}{\times} \theta)_S$ we may now define the following scalar:

$$((\mathbf{AM}, \mathbf{BN}))_S = \mathbf{A} : \mathbf{MB} : \mathbf{N} - \mathbf{A} : \mathbf{NB} : \mathbf{M} \tag{32}$$

where $\mathbf{A}$, $\mathbf{M}$, $\mathbf{B}$, and $\mathbf{N}$ are any four polyadics whatever. If φ and θ are any two double polyadics the scalar $((\varphi, \theta))_S$ is to be found by multiplying the double polyadics term by term and adding the scalars of all the terms. This scalar will be treated later on as a kind of product of the two double polyadics. For the present it will suffice to note that if $\theta = \varphi$ the scalar becomes twice the coefficient of φ^{n-2} in the Hamilton-Cayley equation, or

$$2m_2 = ((\varphi, \varphi))_S. \tag{33}$$

The proof is simple. Let φ be expanded in the form (23), and note that by (24) we have

$$\mathbf{E}_i : \mathbf{M}_k = m_{ki}. \tag{34}$$

Multiplying φ into itself we shall have terms of the form $\mathbf{E}_i\mathbf{M}_i\mathbf{E}_i\mathbf{M}$, whose scalar on developing by (32) vanishes, and terms of the form $\mathbf{E}_i\mathbf{M}_i\mathbf{E}_k\mathbf{M}_k$, whose scalar is $m_{ii}m_{kk} - m_{ik}m_{ki}$ a two-row minor from the determinant of the n^2 elements m_{pq} of φ. Each of the latter terms will occur twice, with all possible selections of pairs of subscripts

Hence the required scalar is twice the sum of two-row minors whose main diagonals coincide with the main diagonal of the determinant of φ. That is, (by the usual matrix theory), it equals $2m_2$, as we see by developing (29).

More generally, let $\mathbf{A}_1, \mathbf{A}_2, \cdots \mathbf{A}_p$ and $\mathbf{B}_1, \mathbf{B}_2, \cdots \mathbf{B}_p$ be any $2p$ polyadics. **Definition.** We define the scalar

$$((\mathbf{A}_1\mathbf{B}_1, \mathbf{A}_2\mathbf{B}_2, \cdots, \mathbf{A}_p\mathbf{B}_p))_S \tag{35}$$

to be the sum of terms computed as follows: the leading term is the product of K-tuple dot products $\mathbf{A}_1 : \mathbf{B}_1\mathbf{A}_2 : \mathbf{B}_2 \cdots \mathbf{A}_p : \mathbf{B}_p$; the other terms are of like form, and are obtained from the leading term by keeping the antecedents $\mathbf{A}_1 \cdots \mathbf{A}_p$ fixed in position while the consequents $\mathbf{B}_1 \cdots \mathbf{B}_p$ are permuted in all ways; the sign of any term is positive or negative according as the number of simple interchanges needed to form the term is even or odd.

If $\varphi_1, \varphi_2, \cdots, \varphi_p$ are any p double polyadics, the scalar

$$((\varphi_1, \varphi_2, \cdots, \varphi_p))_S \tag{36}$$

is defined to be the result of expanding each φ as in (23), multiplying out, and adding the scalars formed from each term by the above definition. We may now suppose these p double polyadics to be all equal. The scalar (36) becomes $((\varphi, \varphi, \cdots \text{ to } p \text{ factors}))_S$ and may be abbreviated $((\varphi^p))_S$. It is now easy to see that

$$((\varphi^p))_S = p!m_p \tag{37}$$

Proof. Expand φ as in (23). The terms of the product $\varphi, \varphi, \cdots$, to p factors are of the form

$$\mathbf{E}_i\mathbf{M}_i, \mathbf{E}_j\mathbf{M}_j, \mathbf{E}_k\mathbf{M}_k, \cdots, \mathbf{E}_r\mathbf{M}_r \tag{38}$$

where subscripts may have any values from 1 to n; we see by (34) that the scalar of this expression has for its leading term $m_{ii}m_{jj}m_{kk} \cdots m_{rr}$, while, by the definition, the other terms are formed by interchanging the subscripts corresponding to the consequents. Therefore the scalar obtained from each expression (38) is a determinant of order p whose main diagonal lies on the main diagonal of the determinant of φ, provided the p subscripts $i, j, k, \cdots, r$ are all different; if they are not all different the scalar of the expression vanishes because it is a determinant with two or more identical rows as well as columns. Each choice of subscripts of non-vanishing terms will occur $p!$ times. The sum of the scalars of all expressions (38) is then $p!$ times the sum

of p- row minors along the main diagonal of the determinant of the m_{ik}. By the usual matrix theory, it is therefore $p!m_p$.

In the above demonstration it is assumed that φ has been expanded in the form (23). It follows from the distributive character of all the steps involved in calculating these scalars that their value is independent of the particular form of φ, precisely as in the case of similar scalars occurring in the usual vector analysis. This is a property of fundamental importance. For, when we have to perform operations into which these quantities enter, it is frequently sufficient to treat a single expression such as (35) in place of the sum of expressions obtained by expansion of (36); and we may develop φ, when necessary, as $\Sigma\mathbf{AM}$ in any desired manner.

To calculate any coefficient m_p in the Hamilton-Cayley equation, we have then merely to express φ as $\Sigma\mathbf{AM}$ in any convenient way, make all possible selections of p different terms, form the scalar (35) for each selection, and add the results, dividing by $p!$.

Many other analogies with ordinary dyadics will naturally suggest themselves. It will be sufficient at this point to note briefly two of the more significant of these.

By solving the Hamilton-Cayley equation as if φ were a scalar we may find roots $g_1, g_2, \cdots, g_n$ and corresponding polyadics $\mathbf{R}_1, \mathbf{R}_2, \cdots, \mathbf{R}_n$ such that

$$(\varphi - g_i I) : \mathbf{R}_i = 0, \quad (i = 1, 2, \cdots, n). \tag{39}$$

The polyadics $\mathbf{R}_1 \cdots \mathbf{R}_n$ may by analogy be called the axes of the double polyadic φ with respect to dot product.

Again, we may say that a double polyadic which can be expressed as the sum of l (and no fewer) properly chosen dyads $\mathbf{AM}$ has $n-l$ degrees of nullity. To get the simplest criterion of the number of degrees of nullity, we may, following Gibbs and Wilson,[6] introduce cross products of polyadics by the law

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{40}$$

and it is then possible to follow reasoning parallel to that of the paper referred to in Note 3, so far as concerns the number of degrees of nullity, by use of double powers of φ. Evidently a product defined by (40) is not itself a *multiple* cross product in the sense that it would reduce to double cross product when $K = 2$. For in Gibbs' system the product by double cross of two dyadics with one common factor vanishes, thus $\mathbf{ab} \times \mathbf{ac} = 0$, while by (40) the cross product of two polyadics vanishes only when they are scalar multiples of one another.

It need hardly be emphasized that formal analogies, when they exist, are of much value, but do not in themselves imply more than certain common laws of operation.

6. The Scalars (36) are Sums of Cubic Determinants.

The quantities denoted by (36) are more general than the coefficients in the Hamilton-Cayley equation. When p has any value from 2 to n such a scalar is a function of all the elements of each of the double polyadics which enter. Suppose $p = 2$ and let φ_1 and φ_2 be expanded in the form (23). The scalar elements of φ_1 and φ_2 may be designated respectively by m_{1ij} and m_{2ij}. We shall have in the indeterminate product $\varphi_1\varphi_2$ pairs of terms of the form $\mathbf{E}_i\mathbf{M}_{1i}$, $\mathbf{E}_j\mathbf{M}_{2j} + \mathbf{E}_j\mathbf{M}_{1j}$, $\mathbf{E}_i\mathbf{M}_{2i}$ which, by forming the scalar by the definition (32), gives $m_{1ii}m_{2jj} - m_{1ij}m_{2ji} + m_{1jj}m_{2ii} - m_{1ji}m_{2ij}$. These four scalar terms are the same as the cubic determinant

$$\begin{vmatrix} m_{1ii}, & m_{1ij} & \vdots & m_{2ii}, & m_{2ij} \\ m_{1ji}, & m_{1jj} & \vdots & m_{2ji}, & m_{2jj} \end{vmatrix} \tag{41}$$

where the indices i and j are signant,[7] but the first index is non-signant as might be expected because $((\varphi_1, \varphi_2))_S = ((\varphi_2, \varphi_1))_S$. We see that if the respective determinants of φ_1 and φ_2 be placed as non-signant layers of a cubic matrix, the cubic determinant (41) has for its own non-signant layers the two-row minors whose main diagonals occupy corresponding positions on the main diagonals of φ_1 and φ_2. As a verification, if $\varphi_1 = \varphi_2$ the cubic determinant does not vanish, but becomes twice one of these minors, in agreement with the preceding discussion of the coefficients in the Hamilton-Cayley equation for φ.

Since i and j have all pairs of values from 1 to n, it is clear that $((\varphi_1\varphi_2))_S$ is the sum of all cubic determinants of the second order whose main diagonals occupy corresponding positions on the main diagonals of the determinants of φ_1 and φ_2.

An analogous proof holds for all values of p up to n. Thus (36) can be written as the sum of all the cubic determinants of order p whose main diagonals are corresponding elements from the determinants of $\varphi_1, \varphi_2, \cdots, \varphi_p$.

In particular, when $p = n$, that is, when we have any n double polyadics, the scalar (36) is the cubic determinant of order n of which the non-signant layers are formed by the square matrices m_{1ij}, m_{2ij}, $\cdots$, m_{nij} of these double polyadics.

7. Scalars formed in Combination with the Idemfactor.

It may happen that one or more of the double polyadics of the scalar (36) is the idemfactor. The scalar formed from any p double polyadics together with the idemfactor taken q times is the same as the scalar formed from the p double polyadics alone, aside from a numerical factor f that is

$$((\varphi_1, \varphi_2, \cdots, \varphi_p, I, I, \cdots))_S = f\,((\varphi_1, \varphi_2, \cdots, \varphi_p))_S \qquad (42)$$

where, however, $p + q$ is not to be greater than n. It is sufficient to prove the proposition for one idemfactor and any p double polyadics, assuming $p + 1$ not greater than n; the general form of (42) then follows by letting some of the polyadics equal the idemfactor. I shall in fact prove the identity for one idemfactor with $p = 2$; the proof for other values of p is quite similar. The scalar $((\varphi_1, \varphi_2, I))_S$ is by Art 6 the sum of all cubic determinants of the third order of the form

$$\begin{vmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{vmatrix} \cdot \begin{vmatrix} m_{1ii}, & m_{1ij}, & m_{1ik} \\ m_{1ji}, & m_{1jj}, & m_{1jk} \\ m_{1ki}, & m_{1kj}, & m_{1kk} \end{vmatrix} \cdot \begin{vmatrix} m_{2ii}, & m_{2ij}, & m_{2ik} \\ m_{2ji}, & m_{2jj}, & m_{2jk} \\ m_{2ki}, & m_{2kj}, & m_{2kk} \end{vmatrix} \qquad (43)$$

which on development yields the sum of three cubic determinants of the second order having the same form as (41), and corresponding to the three pairs of subscripts ij, jk, ki. Consider some particular one of these cubic determinants of the second order, say that one which involves the subscripts ij. It will evidently occur once in the expansion of every cubic determinant of the form (43), provided the subscripts are i, j and any other subscript not greater than n. That is, the cubic determinant (41) will occur $n - 2$ times in the development of $((\varphi_1, \varphi_2, I))_S$. But i and j are any pair of different subscripts. Hence

$$((\varphi_1, \varphi_2, I))_S = (n - 2)\,((\varphi_1, \varphi_2))_S \qquad (44)$$

The same reasoning applies for any value of p. Hence we have

$$((\varphi_1, \varphi_2, \cdots, \varphi_p, I))_S = (n - p)\,((\varphi_1, \varphi_2, \cdots, \varphi_p))_S \qquad (44a)$$

where p has any value from 1 to $n - 1$ and $\varphi_1 \cdots \varphi_p$ are any double K-adics whatever. By letting several of these K-adics successively become equal to the idemfactor we obtain (42) and see that the numerical factor f is given by

$$f = (n - p - q + 1)\,(n - p - q + 2) \cdots (n - p) \qquad (45)$$

We might also have obtained (44) by a count of the number of terms obtained by developing either side. If ${}_nC_{p+1}$ denote the number of combinations of n things $p+1$ at a time, this will be the number of cubic determinants of order $p+1$ obtained on developing the left side of (44). Each of these cubic determinants yields $p+1$ cubic determinants of order p, because one layer is taken from the idemfactor, hence contains $p+1$ non-vanishing elements. Thus the left side of (44) may be written as the sum of $(p+1){}_nC_{p+1}$ cubic determinants of order p. On the right the scalar $((\varphi_1, \varphi_2, \cdots, \varphi_p))_S$ is the sum of ${}_nC_p$ cubic determinants of order p. But we have identically

$$(p+1){}_nC_{p+1} = (n-p){}_nC_p \tag{46}$$

agreeing with (44).

In particular it follows from (42) that any scalar (36) can be written as a single cubic determinant of order n by adjoining the idemfactor $n-p$ times, so that $p+q=n$ and the numerical factor becomes $(n-p)!$. Thus

$$((\varphi_1, \varphi_2 \cdots, \varphi_p, I, I, \cdots))_S = (n-p)!\ ((\varphi_1, \varphi_2 \cdots, \varphi_p))_S \tag{47}$$

in the case where the idemfactor occurs just $n-p$ times.

It also follows that any coefficient in the Hamilton-Cayley equation for φ can be written as a single cubic determinant of order n. For by letting the p arbitrary double polyadics all equal φ, we have in virtue of (37)

$$((\varphi, \varphi, \cdots, \varphi, I, I, \cdots I))_S = p!\ (n-p)!\ m_p \tag{48}$$

provided φ enters p times and I enters $n-p$ times.

8. Scalars Expressed as Square Determinants whose Elements are K-adics.

By virtue of the result of Art. 7, it becomes possible to express any scalar of the form (36) without the use of cubic determinants, if we are willing instead to use determinants whose elements are not scalars but polyadics, and to make certain slight but needful modifications in the usual rules for determinate expansions.

Suppose, as before, that there are n arbitrary double polyadics $\varphi_1 \cdots \varphi_n$ and let each of them be expanded in the manner of (27), thus

$$\varphi_i = \mathbf{B}_{i1}\mathbf{E}_1 + \mathbf{B}_{i2}\mathbf{E}_2 + \cdots + \mathbf{B}_{in}\mathbf{E}_n \tag{49}$$

where the consequents $\mathbf{E}_1 \cdots \mathbf{E}_n$ are the fundamental K-ads and the antecedents are any arbitrary K-adics. Let $\mathbf{E}_j$ be any one of the fundamental K-ads. We have

$$\varphi_i : \mathbf{E}_j = \mathbf{B}_{ij}. \tag{50}$$

The binary assemblage of n^2 arbitrary polyadics $\mathbf{B}_{ij}$ are now to be formed into a determinant

$$\begin{vmatrix} \mathbf{B}_{11}, & \mathbf{B}_{12}, & \cdots, & \mathbf{B}_{1n} \\ \mathbf{B}_{21}, & \mathbf{B}_{22}, & \cdots, & \mathbf{B}_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mathbf{B}_{n1}, & \mathbf{B}_{n2}, & \cdots, & \mathbf{B}_{nn} \end{vmatrix} \tag{51}$$

where the law of expansion is defined to be as follows: the leading term is written $[\mathbf{B}_{11}\mathbf{B}_{22} \cdots \mathbf{B}_{nn}]$ and signifies the determinant in the ordinary sense whose elements are the scalar elements of the polyadics which enter: namely, if we expand as

$$\mathbf{B}_{ij} = b_{ij1}\mathbf{E}_1 + b_{ij2}\mathbf{E}_2 + \cdots + b_{ijn}\mathbf{E}_n \tag{52}$$

the leading term in expanding (51) is the determinant of ordinary character

$$\begin{vmatrix} b_{111}, & b_{112}, & \cdots, & b_{11n} \\ b_{221}, & b_{222}, & \cdots, & b_{22n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ b_{nn1}, & b_{nn2}, & \cdots, & b_{nnn} \end{vmatrix} \tag{53}$$

the other terms in the expansion of (51) are of similar character, and are formed by writing out the determinant (51) as if the elements were scalars, enclosing each term in square brackets to signify the determinant formed from it as in (53). The signs will occur as in ordinary determinants if we adopt the rule that, in every term, *the order of the polyadics shall be the order of the rows in the original determinant.* The meaning of the determinant (51) is thus defined without ambiguity. We note that by this rule the order of rows in (53) corresponds to the order of rows in (51), in the sense that each row of (53) consists of the elements of a single polyadic from the corresponding row of (51).

It is then apparent that (51) is the sum of $n!$ ordinary determinants, hence the sum of $(n!)^2$ terms on complete expansion. The order of the rows is non-signant, in the sense that, if two rows in (51) be interchanged, the value of the expression is quite unchanged; for on ex-

panding, first, into a sum of ordinary determinants or square bracket expressions, the sign before every square bracket is changed because the signs behave as if (51) were an ordinary determinant; but, since the development follows the rows, the sign of every determinant (53) will have been changed due to the interchange of rows; thus each square bracket has its sign changed, we may say, both from within and from without. Hence the whole expression is unaltered. On the other hand, an interchange of two columns in (51) changes the sign before every square bracket but leaves the brackets themselves unaffected, for the development is according to rows; hence the sign of the whole is changed.

It is now not hard to see that the polyadic determinant (51) is the same as $((\varphi_1, \varphi_2, \cdots, \varphi_n))_S$. For we have seen that the first index is non-signant, the second signant; the third is obviously signant since it governs the columns of the ordinary determinants of the form (53) but does not otherwise enter. Furthermore, on complete expansion of (51) we shall have each term a product of n elements, one from each of the double polyadics, one from each row of (51), and one from each column of (51). But this is precisely the law of formation of a cubic determinant having the elements of the polyadics as non-signant layers. By Art. 6 this is the scalar in question and we may write the identity

$$\begin{vmatrix} \varphi_1 : \mathbf{E}_1, & \varphi_1 : \mathbf{E}_2, & \cdots, & \varphi_1 : \mathbf{E}_n \\ \varphi_2 : \mathbf{E}_1, & \varphi_2 : \mathbf{E}_2, & \cdots, & \varphi_2 : \mathbf{E}_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi_n : \mathbf{E}_1, & \varphi_n : \mathbf{E}_2, & \cdots, & \varphi_n : \mathbf{E}_n \end{vmatrix} = ((\varphi_1, \varphi_2, \cdots, \varphi_n))_S \qquad (54)$$

where the polyadic determinant on the left is defined as above.

To take the simplest possible illustration, let $K = 1$, $N = 2$, $n = 2$. Our K-adics become vectors in two dimensions. Take $\mathbf{E}_1 = \mathbf{i}$, $\mathbf{E}_2 = \mathbf{j}$. Let there be two dyadics $\varphi_1 = \mathbf{b}_{11}\mathbf{i} + \mathbf{b}_{12}\mathbf{j} = b_{111}\mathbf{ii} + b_{112}\mathbf{ji} + b_{121}\mathbf{ij} + b_{122}\mathbf{jj}$ and similarly for φ_2. The determinant on the left of (54) becomes

$$\begin{vmatrix} b_{111}\mathbf{i} + b_{112}\mathbf{j}, & b_{121}\mathbf{i} + b_{122}\mathbf{j} \\ b_{211}\mathbf{i} + b_{212}\mathbf{j}, & b_{221}\mathbf{i} + b_{222}\mathbf{j} \end{vmatrix}$$

whose elements are vectors. Expanding by the rule we have

$$[b_{111}\mathbf{i} + b_{112}\mathbf{j}, b_{221}\mathbf{i} + b_{222}\mathbf{j}] - [b_{121}\mathbf{i} + b_{122}\mathbf{j}, b_{211}\mathbf{i} + b_{212}\mathbf{j}]$$

which is the same as the difference of two determinants

$$(b_{111}b_{222} - b_{112}b_{221}) - (b_{121}b_{212} - b_{122}b_{211})$$

and this, again, agrees with the cubic determinant formed with the element of the two dyadics as layers, namely

$$\begin{vmatrix} b_{111}, & b_{121} \\ b_{112}, & b_{122} \end{vmatrix} \quad \begin{vmatrix} b_{211}, & b_{221} \\ b_{212}, & b_{222} \end{vmatrix}$$

A number of results follow immediately. We might have formed the left side of (54) by taking the multiple dot product of $\mathbf{E}_1$, $\mathbf{E}_2$, etc. into each of the arbitrary double polyadics, as $\mathbf{E}_1 : \varphi_1$, $\mathbf{E}_2 : \varphi_1$, etc. in the first row, and similarly for the other rows. Instead of (51) we should have had the determinant of the $\mathbf{M}_{ij}$ by developing after the manner (23) instead of (27). The rest of our reasoning would have been unchanged, however, hence the value of the expression would have remained the same. The change would in fact be equivalent to changing rows into columns and columns into rows in all the non-signant layers of the cubic determinant, which, since the second and third indices are signant, leaves the result unchanged. This is illustrated by the example just given, where the last index refers to the rows instead of to the columns as in (41). If we let φ' stand for $\Sigma\mathbf{MA}$ when φ stands for $\Sigma\mathbf{AM}$, and say that the conjugate φ' of a double polyadic φ is formed from it by interchanging antecedent and consequent, it is clear that $\mathbf{E} : \varphi = \varphi' : \mathbf{E}$. We therefore have the identity

$$((\varphi_1, \varphi_2, \cdots, \varphi_n))_S = ((\varphi_1', \varphi_2', \cdots, \varphi_n'))_S \tag{55}$$

which may also be deduced directly from the definition of these scalars. The well known fact that the Hamilton-Cayley equation for any dyadic or matrix is identical with that of its conjugate appears as a special case of this identity.

Also if $n - p$ out of the arbitrary double polyadics are allowed to become equal to the idemfactor, the identity (54) becomes in virtue of (47),

$$\begin{vmatrix} \varphi_1 : \mathbf{E}_1, & \varphi_1 : \mathbf{E}_2, & \cdots, & \varphi_1 : \mathbf{E}_n \\ \varphi_2 : \mathbf{E}_1, & \varphi_2 : \mathbf{E}_2, & \cdots, & \varphi_2 : \mathbf{E}_n \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_p : \mathbf{E}_1, & \varphi_p : \mathbf{E}_2, & \cdots, & \varphi_p : \mathbf{E}_n \\ \mathbf{E}_1, & \mathbf{E}_2, & \cdots, & \mathbf{E}_n \\ \mathbf{E}_1, & \mathbf{E}_2, & \cdots, & \mathbf{E}_n \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{E}_1, & \mathbf{E}_2, & \cdots, & \mathbf{E}_n \end{vmatrix} = (n-p)!((\varphi_1, \varphi_2, \cdots, \varphi_p))_S \tag{56}$$

where the last $n - p$ rows are alike, a circumstance which need cause no surprise, since, as already pointed out, the rows are non-signant in the very nature of the definition of these polyadic determinants.[8]

If we still further specialize (56) by letting the remaining p double polyadics all become equal to φ we shall have, by (48),

$$\begin{vmatrix} \varphi:\mathbf{E}_1, & \varphi:\mathbf{E}_2, & \cdots, & \varphi:\mathbf{E}_n \\ \varphi:\mathbf{E}_1, & \varphi:\mathbf{E}_2, & \cdots, & \varphi:\mathbf{E}_n \\ \cdots & \cdots & \cdots & \cdots \\ \varphi:\mathbf{E}_1, & \varphi:\mathbf{E}_2, & \cdots, & \varphi:\mathbf{E}_n \\ \mathbf{E}_1, & \mathbf{E}_2, & \cdots, & \mathbf{E}_n \\ \mathbf{E}_1, & \mathbf{E}_2, & \cdots, & \mathbf{E}_n \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{E}_1, & \mathbf{E}_2, & \cdots, & \mathbf{E}_n \end{vmatrix} = p!\,(n-p)!\,m_p \qquad (57)$$

where, on the left, p rows are like the first row, and $(n - p)$ rows are like the last row.

9. An Invariant Property of the Scalars (36).

From the form of the definition of the scalars of the present discussion, they are independent of development in terms of a particular set of polyadics. Throughout the argument the fundamental polyads $\mathbf{E}_1 \cdots \mathbf{E}_n$ may be replaced by any other set of n polyadics forming a normal, orthogonal system with respect to K-tuple dot product, for no other properties of these polyads have been so far used. These scalars have another kind of invariance of a somewhat wider sort, with respect to a reference system which need no longer be normal nor orthogonal but only linearly independent. The properties in question are essentially included in the following theorem: —

Theorem. The scalar $((\varphi_1, \varphi_2, \cdots, \varphi_n))_S$ formed from a set of n arbitrary double polyadics is unaltered when each of these double polyadics is multiplied by (or into) the same double polyadic θ, (by K-tuple dot product), except for a factor which is the determinant of θ.

This theorem is equivalent to the identity

$$((\varphi_1, \varphi_2, \cdots, \varphi_n))_S\,((\theta^n))_S = n!\,((\varphi_1:\theta, \varphi_2:\theta, \cdots, \varphi_n:\theta))_S \qquad (58)$$

for, by (37), $((\theta^n))_S$ equals $n!$ times the coefficient of θ^n in the Hamilton-Cayley equation for θ, which is the determinant of θ; and $\theta:\varphi_1 = \varphi_1':\theta$, while by (55) the left side of (58) is unchanged if we change every φ into φ'; hence it is sufficient to prove (58) as written.

Proof. Let every φ be expanded as in (23). The scalar $((\varphi_1 \cdots, \varphi_n))_S$ is equal to the polyadic determinant of the $\mathbf{M}_{ij}$ as shown in Art. 8. To multiply every φ into a double polyadic θ is equivalent to performing the same linear transformation upon every $\mathbf{M}$. But the polyadic determinant is the sum of ordinary determinants whose rows consist of elements of the respective polyadics $\mathbf{M}$. It is well known that when every row of a determinant is affected by the same linear transformation the determinant is merely multiplied by the determinant of the transformation. Thus every square bracket term in the expansion of the polyadic determinant of the $\mathbf{M}_{ij}$ is multiplied by the determinant of θ. The theorem is therefore proved.

As an immediate consequence, any polyadic element $\varphi_i : \mathbf{E}_j$ on the left of the identity (54) may be replaced by $\varphi_i : \theta : \mathbf{E}_j$, provided we multiply the right side by the determinant of θ, which may be called m_θ and is the same as the determinant of the n polyadics $\theta : \mathbf{E}_1$, $\theta : \mathbf{E}_2 \cdots, \theta : \mathbf{E}_n$. By a proper choice of the arbitrary double polyadic θ, these n polyadics may be made to be any n polyadics whatever, $\mathbf{A}_1, \cdots, \mathbf{A}_n$. If we agree to write $[\varphi_i : \mathbf{A}_j]$ for the polyadic determinant whose elements have the form indicated, and $[a_{ij}]$ for the determinant (in the ordinary sense), whose elements are the scalar elements of the polyadics $\mathbf{A}_1 \cdots \mathbf{A}_n$ resolved according to (11), we shall now have the identity

$$((\varphi_1, \varphi_2, \cdots, \varphi_n))_S = \frac{[\varphi_i : \mathbf{A}_j]}{[a_{ij}]} \tag{59}$$

assuming the polyadics $\mathbf{A}_i$ linearly independent. Thus the scalar on the left is invariant of the particular choice of these polyadics.[9]

In a similar manner and with similar notation we may derive from (56)

$$[\varphi_1 : \mathbf{A}_j, \varphi_2 : \mathbf{A}_j, \cdots, \varphi_p : \mathbf{A}_j, \mathbf{A}_j, \mathbf{A}_j, \cdots, \mathbf{A}_j] = [a_{ij}]\,(n-p)!\,((\varphi_1, \varphi_2, \cdots, \varphi_p))_S \tag{60}$$

and from (57)

$$[(\varphi : \mathbf{A}_j)^p\,(\mathbf{A}_j)^{n-p}] = [a_{ij}]\,p!\,(n-p)!\,m_p \tag{61}$$

In employing this abbreviated notation for polyadic determinants, if there are two subscripts the first always refers to the rows, the second to the columns, as in (59) and (60). Where a series of n polyadics is written in brackets as in (60) and (61) these refer to the rows. We pick out an element of a particular column by giving a particular value

to the common subscript j. If several rows are alike the fact may be denoted by an exponent, as in (61). Thus in (60) the left side might have been written $[\varphi_1 : \mathbf{A}_j, \cdots, \varphi_p : \mathbf{A}_j, (\mathbf{A}_j)^{n-p}]$. Where, however, the notation implies merely n polyadics (instead of n^2) the square bracket signifies the ordinary determinant of the n^2 elements by the scheme of (11). Thus instead of $[a_{ij}]$ we might write $[\mathbf{A}_j]$ without change of meaning. Unless the contrary is stated, all subscripts are to run from 1 to n.

10. Forms of the Scalars (36) which show their Polyadic Character.

Up to this point have been noted three distinct methods of actually calculating a scalar of the form (36), namely

1°. Multiply the double polyadics term by term and take scalars according to definition.

2°. Form a sum of cubic determinants.

3°. Form a polyadic determinant by operating on any set of linearly independent polyadics and divide by the determinant of the set.

In all these cases the double polyadics have behaved formally as if their antecedents and consequents were vectors in space of n dimensions, instead of polyadics each of which is a sum of products of K vectors in space of N dimensions, where $n = N^K$. Each of these methods applies with no change whatever if $K = 1$ so that $n = N$ and the polyadics become actually vectors,— with the obvious exception that the multiple dot product, indicated by a colon, becomes ordinary or single dot product. While I am not aware that any of these three methods has actually been employed by writers on vector analysis in N dimensions, yet it can hardly be said that the individual steps do not occur, in principle at least, in the writings of Hamilton, Gibbs, or Grassmann. For example, the definition of these scalars is equivalent to a double indeterminate product followed by inner product. In the case $K = 1$, $N = 3$ they were introduced by Gibbs as scalars of double cross products. In the same case they were expressed by Hamilton both by summations which are equivalent to double multiplication,[10] (though never so regarded by him), and also as quotients equivalent to our method 3°. It is true that none of these writers expressed the scalars as sums of determinants whose elements are extensive magnitudes, use of which was first made by Cayley, nor by cubic determin-

ants, also due to Cayley, but these two equivalent formulations differ more in appearance than in reality from such expressions as Hamilton's

$$-m_2 = Si\varphi j\varphi k + Si\varphi k\varphi i + Sk\varphi i\varphi j$$

which in Gibbs' notation would be

$$m_2 = \Sigma(\mathbf{i}\varphi\cdot\mathbf{j}\varphi\cdot\mathbf{k})$$

where i, j, k denote rectangular unit vectors.

It will be of some value for the sake of later applications if we take account of these scalars as they appear when our polyadics are determined by K-dimensional matrices. Consider for a moment the case $((\mathbf{AM}, \mathbf{BN}))_S$ where **A**, **M**, **B**, and **N** are dyadics in two dimensions,— evidently the simplest case which can be regarded as a generalization of the usual vector analysis. With unit vectors **i** and **j** we may either write

$$\mathbf{A} = a_{11}\mathbf{ii} + a_{12}\mathbf{ij} + a_{21}\mathbf{ji} + a_{22}\mathbf{jj} \tag{62}$$

or may regard **A** as an abbreviation for the matrix

$$\left\| \begin{matrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{matrix} \right\| \tag{63}$$

with similar notation for the other three dyadics. If we form the required scalar directly from the definition, namely as $\mathbf{A:MB:N} - \mathbf{A:NB:M}$ we obtain

$$\begin{aligned}((\mathbf{AM}, \mathbf{BM}))_S = {} & (a_{11}m_{11} + a_{12}m_{12} + a_{21}m_{21} + a_{22}m_{22})\,(b_{11}n_{11} + b_{12}n_{12} \\ & + b_{21}n_{21} + b_{22}n_{22}) \\ & - (a_{11}n_{11} + a_{12}n_{12} + a_{21}n_{21} + a_{22}n_{22})\,(b_{11}m_{11}b_{12}m_{12} + \\ & b_{21}m_{21} + b_{22}m_{22})\end{aligned} \tag{64}$$

If instead we use the method of cubic determinants we shall first form the square array of the fourth order belonging to the double dyadic **AM**, which is

$$\begin{matrix} a_{11}m_{11}, & a_{11}m_{12}, & a_{11}m_{21}, & a_{11}m_{22} \\ a_{12}m_{11}, & a_{12}m_{12}, & a_{12}m_{21}, & a_{12}m_{22} \\ a_{21}m_{11}, & a_{21}m_{12}, & a_{21}m_{21}, & a_{21}m_{22} \\ a_{22}m_{11}, & a_{22}m_{12}, & a_{22}m_{21}, & a_{22}m_{22} \end{matrix} \tag{65}$$

with a similar array for **BN**. We then take the sum of all cubic determinants along the main diagonal, using the two arrays as similarly placed layers. The first of these will be

$$\begin{vmatrix} a_{11}m_{11}, & a_{11}m_{12} \\ a_{12}m_{11}, & a_{12}m_{12} \end{vmatrix} \begin{vmatrix} b_{11}n_{11}, & b_{11}n_{12} \\ b_{12}n_{11}, & b_{12}n_{12} \end{vmatrix} \tag{66}$$

which develops into

$$a_{11}m_{11}b_{12}n_{12} - a_{12}m_{11}b_{11}n_{12} - a_{11}m_{12}b_{12}n_{11} + a_{12}m_{12}b_{11}n_{11} \tag{67}$$

There are six of these groups of terms, or 24 in all. The scheme of each group is given by

$$\begin{vmatrix} a_{pq}m_{pq}, & a_{pq}m_{rs} \\ a_{rs}m_{pq}, & a_{rs}m_{rs} \end{vmatrix} \begin{vmatrix} b_{pq}n_{pq}, & b_{pq}n_{rs} \\ b_{rs}n_{pq}, & b_{rs}n_{rs} \end{vmatrix} \tag{68}$$

which develops into

$$a_{pq}m_{pq}b_{rs}n_{rs} - a_{rs}m_{pq}b_{pq}n_{rs} - a_{pq}m_{rs}b_{rs}n_{pq} + a_{rs}m_{rs}b_{pq}n_{pq} \tag{69}$$

and the summation is performed by giving to the number-pairs pq and rs unlike values chosen from the pairs 11, 12, 21, and 22. Comparing with (64) it is easy to verify that the two methods agree.

If we use the method of polyadic determinants and adopt the dyads **ii**, **ij**, etc. as operands we first form the dyadics **AM : ii**, **AM : ij**, etc. treating **BN** in a similar manner. We thus have two rows of a determinant of the fourth order whose elements are dyadics. The other two rows are alike and given by I : **ii**, I : **ij**, etc. that is by **ii**, **ij**, etc. The result

$$\begin{vmatrix} \mathbf{A}m_{11}, & \mathbf{A}m_{12}, & \mathbf{A}m_{21}, & \mathbf{A}m_{22} \\ \mathbf{B}n_{11}, & \mathbf{B}n_{12}, & \mathbf{B}n_{21}, & \mathbf{B}n_{22} \\ \mathbf{ii} \ , & \mathbf{ij} \ , & \mathbf{ji} \ , & \mathbf{jj} \\ \mathbf{ii} & \mathbf{ij} & \mathbf{ji} & \mathbf{jj} \end{vmatrix} \tag{70}$$

should by (56) be the double of the required scalar. Developing by the definition of Art. 8 the leading term is $[\mathbf{A}m_{11}, \mathbf{B}n_{12}, \mathbf{ji}, \mathbf{jj}]$ which denotes the ordinary determinant of the fourth order

$$\begin{vmatrix} a_{11}m_{11}, & a_{12}m_{11}, & a_{21}m_{11}, & a_{22}m_{11} \\ b_{11}n_{12} \ , & b_{12}n_{12} \ , & b_{21}n_{12} \ , & b_{22}n_{22} \\ 0 \ , & 0 \ , & 1 \ , & 0 \\ 0 \ , & 0 \ , & 0 \ , & 1 \end{vmatrix} \tag{71}$$

which develops into $a_{11}m_{11}b_{12}n_{12} - a_{12}m_{11}b_{11}n_{12}$. The next term may be taken as $-[\mathbf{A}m_{11}, \mathbf{B}n_{12}, \mathbf{jj}, \mathbf{ji}]$ which doubles the two scalar terms already found. It is easy to see that developing (70) by two-row minors after Laplace's method yields the scalar terms in a fashion similar to the method of cubic determinants.

The fact that any scalar (36) is unaltered by changing all the double dyadics into their conjugates, and is also unaltered by changing the order of double dyadics, appears in the double symmetry of (64). A further and quite distinct fact could not have been brought out so long as our notation represented polyadics as formal vectors: a scalar formed from double dyadics, like (64), is unaltered if all the dyadics **A**, **M**, **B**, **N** are replaced by their conjugates. This is equivalent to interchanging the two subscripts throughout the expression on the right of (64); whereas changing the double polyadics into their conjugates is equivalent to everywhere interchanging a with m and at the same time b with n; and changing the order of the polyadics is equivalent to everywhere interchanging a with b and at the same time m with n.

More generally a polyadic which is a sum of terms of the form $\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_k$ where the K factors are vectors in space of N dimensions may be regarded as one of a set of $K!$ polyadics obtained from one another by making like permutations of factors in every term. A glance at the definition (10) of multiple dot product is sufficient to show that all scalars (36) are unaltered when all the antecedents and consequents of every double polyadic have the factors of all their terms permuted in the same manner. If each polyadic consists of a single polyad and we apply the definition of Art. 5, the truth of the proposition is evident, hence by the distributive principle holds universally.

Examples like (64) or (69) may be generalized in three ways, according as we increase N, the dimensions of the space, K, the order of the polyadics, or p, the order of the scalars.

If N increases we still have double dyadics and the form of (69) is unaltered, but the summation is performed over a larger group of number-pairs, namely 11, 12, 21, 13, 31,$\cdots$, NN.

If K increases the number of subscripts increases but (69) still consists of 4 scalar terms and the summation is over pairs of unlike number-triplets, quadruplets, etc.

If p increases the order of the determinant (68) increases, likewise the number of terms in (69). Thus if we keep $N = 2$, $K = 2$, but make $p = 3$ there are 4 cubic determinants whose sum is $((\mathbf{AL}, \mathbf{BM}, \mathbf{CN}))_S$ or 72 scalar terms in all.

If we have double dyadics in 3 dimensions with $p = 2$, $((\mathbf{AM}, \mathbf{BN}))_S$ will be the sum of 36 expressions of the same form as (69) or 144 scalar terms.

It is usually possible to neglect the dimensionality of the space and

the number of terms in the double polyadic, frequently also the order of the polyadics. Thus $((\mathbf{AM}, \mathbf{BN}))_S$ is symbolic of $((\varphi_1, \varphi_2))_S$. When the order of the polyadic is given we may also treat with polyads in many operations as symbolic of polyadics. Thus if we have double dyadics in any space we may often write $\mathbf{aa}'$ for $\mathbf{A}$, $\mathbf{mm}'$ for $\mathbf{M}$, etc. in which case a bar may be used to separate antecedent from consequent in the double polyads, or $((\mathbf{aa}' \mid \mathbf{mm}', \mathbf{bb}' \mid \mathbf{nn}'))_S$ may be taken symbolically for the much more general $((\varphi_1, \varphi_2))_S$. Hence if φ_1 and φ_2 are double dyadics we may write

$$((\varphi_1, \varphi_2))_S = \Sigma(\mathbf{a}\cdot\mathbf{ma}'\cdot\mathbf{m}'\mathbf{b}\cdot\mathbf{nb}'\cdot\mathbf{n}' - \mathbf{a}\cdot\mathbf{na}'\cdot\mathbf{n}'\mathbf{b}\cdot\mathbf{mb}'\cdot\mathbf{m}') \quad (72)$$

and may even omit the summation sign in many operations. It is of interest to compare this vector expression with (64) which was based on the notion of the dyadic as a matrix.

11. Invariants regarded as Products.

The examples of the last article are sufficient to indicate that the scalars (36) are invariants of polyadic systems and that they are subject to a considerable variety of transformations. There is one more method by which they may be calculated, based on a concept of these scalars which may serve to widen considerably our view of the formal laws to which double polyadics are amenable. Returning to the earlier conception of a double polyadic as a sum of unit double polyads $\mathbf{E}_i\mathbf{E}_k$ each multiplied by a scalar, as in (25), consider scalars of the form $((\mathbf{E}_i\mathbf{E}_k, \mathbf{E}_r\mathbf{E}_s))_S$, that is $\mathbf{E}_i{:}\,\mathbf{E}_k\mathbf{E}_r{:}\,\mathbf{E}_s - \mathbf{E}_i{:}\,\mathbf{E}_s\mathbf{E}_r{:}\,\mathbf{E}_k$. The unit polyads $\mathbf{E}_i$ and $\mathbf{E}_r$ are antecedents while $\mathbf{E}_k$ and $\mathbf{E}_s$ are consequents. We see that the scalar is equal to $+1$ when each antecedent is equal to its own consequent, but unlike the other antecedent and consequent; is equal to -1 when each antecedent differs from its own consequent but is equal to the other consequent; and is otherwise zero. That is

$$\begin{gathered}((\mathbf{E}_i\mathbf{E}_i, \mathbf{E}_k\mathbf{E}_k))_S = +1,\ ((\mathbf{E}_i\mathbf{E}_k, \mathbf{E}_k\mathbf{E}_i))_S = -1,\ ((\mathbf{E}_i\mathbf{E}_i, \mathbf{E}_r\mathbf{E}_s))_S = 0 \\ ((\mathbf{E}_i\mathbf{E}_i, \mathbf{E}_i\mathbf{E}_i))_S = 0, \quad ((\mathbf{E}_i\mathbf{E}_k, \mathbf{E}_r\mathbf{E}_r))_S = 0, \quad ((\mathbf{E}_i\mathbf{E}_i, \mathbf{E}_i\mathbf{E}_k))_S = 0 \\ ((\mathbf{E}_i\mathbf{E}_k, \mathbf{E}_k\mathbf{E}_k))_S = 0, \quad ((\mathbf{E}_i\mathbf{E}_k, \mathbf{E}_r\mathbf{E}_s))_S = 0\end{gathered} \quad (73)$$

on the understanding that subscripts are unequal unless denoted by the same letter.

In the same way it is easy to see that any scalar derived from a set of p unit double polyads, namely

$$((\mathbf{E}_a\mathbf{E}_i, \mathbf{E}_b\mathbf{E}_k, \cdots, \mathbf{E}_g\mathbf{E}_s))_S \tag{74}$$

where subscripts run from 1 to n, will vanish unless, by interchanges among the consequents, it can be brought to the form

$$((\mathbf{E}_a\mathbf{E}_a, \mathbf{E}_b\mathbf{E}_b, \cdots, \mathbf{E}_g\mathbf{E}_g))_S \tag{75}$$

wherein each antecedent is equal to its own consequent; and moreover vanishes unless all antecedents are unequal. The sign is plus or minus according as the number of interchanges needed to give the form (75) is even or odd.

We may now regard the scalars (36) as products of double polyadics formed according to this law, and, if each double polyadic is developed as in (25), can evidently calculate these scalars with no more labor than would be needed to multiply algebraic polynomials.

12. Star Products.

It is especially in the case $p = 2$ that the concept of the last article appears useful. In keeping with the idea that $((\varphi_1, \varphi_2))_S$, the calculation of which has been exemplified in some detail, is a kind of product of the two double polyadics, we may introduce the notation

$$((\varphi_1, \varphi_2))_S = \varphi_1{}^*\varphi_2 \tag{76}$$

and speak of this scalar as the star product of the two double polyadics. It is essentially a kind of *double* product, but since there is nothing analogous for mere vector factors it is not necessary to write two stars to emphasize the distinction. It may properly be compared, or rather contrasted, with the double dot product of Gibbs. To illustrate, take again the simplest case, $K = 1$, $N = 2$, so that the factors are dyadics in space of two dimensions. By Art. 11 the multiplication table for unit dyads will be

	ii,	**ij**,	**ji**,	**jj**
ii	0,	0,	0,	+1
ij	0,	0,	−1,	0
ji	0,	−1,	0,	0
jj	+1,	0,	0,	0

(77)

which brings out the "skew" character of the star product: it is formed by the multiplication of unlike elements. If the scalar elements of φ_1 and φ_2 are a_{rs} and b_{rs} in the style of (63), we shall have

$$\begin{aligned}\varphi_1{}^*\varphi_2 &= (a_{11}\mathbf{ii} + a_{12}\mathbf{ij} + a_{21}\mathbf{ji} + a_{22}\mathbf{jj})^*(b_{11}\mathbf{ii} + b_{12}\mathbf{ij} + b_{21}\mathbf{ji} + b_{22}\mathbf{jj}) \\ &= a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} \qquad (78)\end{aligned}$$

The result is of course the same as if the dyadics had been expanded in any other manner and we had applied one of the three methods of Art. 10.

By contrast the double dot product $\varphi_1 : \varphi_2$ yields a multiplication table having $+1$ along the main diagonal and elsewhere zero, giving

$$\varphi_1 : \varphi_2 = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} \qquad (79)$$

The concept of $\varphi_1{}^*\varphi_2$ as a product is justified by the fact that it is in every case possible to set up systems of double polyadics φ_1, $\varphi_2, \cdots, \varphi_\nu$ such that $\varphi_r{}^*\varphi_s = 1$ if subscripts are equal, otherwise zero. Here $\nu = n^2 = N^{2K}$ for this is the number of linearly independent double polyadics in any case. In the example (77) if we make

$$\mathbf{F}_1 = \frac{(\mathbf{ii} + \mathbf{jj})}{\sqrt{+2}},\ \mathbf{F}_2 = \frac{(\mathbf{ii} - \mathbf{jj})}{\sqrt{-2}},\ \mathbf{F}_3 = \frac{(\mathbf{ij} + \mathbf{ji})}{\sqrt{-2}},\ \mathbf{F}_4 = \frac{(\mathbf{ij} - \mathbf{ji})}{\sqrt{+2}} \qquad (80)$$

we shall have $\mathbf{F}_r{}^*\mathbf{F}_s = 1$ if subscripts are equal, otherwise zero.

It is clear that, just as the dyads **ii**, **ij**, **ji**, **jj** could be formally treated with reference to double dot product as would be vectors in 4-space with reference to ordinary dot product, so the dyadics **F** can be similarly treated with respect to star product.

A correspondence may be thus set up between invariant properties of the dyadics or polyadics or double polyadics and the geometry of the space in which these magnitudes behave formally like vectors.

A special case of much interest has been treated in great detail, with some differences of terminology, by E. Waelsch.[11] By his very ingenious system the whole of the well-worked theory of invariants of binary forms is employed as algorism for phenomena of various sorts in ordinary space. In the language of the present discussion his system would arise by making up three fundamental units

$$\mathbf{E}_1 = \mathbf{ii},\ \mathbf{E}_2 = \frac{(\mathbf{ij} + \mathbf{ji})}{\sqrt{2}},\ \mathbf{E}_3 = \mathbf{jj} \qquad (81)$$

so that $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ form a normal orthogonal system with respect to double dot product. A binary quadric may be written

$$\alpha_{11}x_1^2 + 2\,\alpha_{12}x_1x_2 + \alpha_{22}x_2^2 \tag{82}$$

and a variable vector may be written

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} \tag{83}$$

but this vector is regarded as merely formal. We have identically

$$[\alpha_{11}\mathbf{ii} + \alpha_{12}(\mathbf{ij} + \mathbf{ji}) + \alpha_{22}\mathbf{jj}]\colon \mathbf{xx} = \alpha_{11}x_1^2 + 2\alpha_{12}x_1x_2 + \alpha_{22}x_2^2 \tag{84}$$

so that any binary quadric is the double dot product of a dyadic into the dyad **xx**. We now make up the new system, orthogonal with respect to star product.

$$\mathbf{F}_1 = \frac{\mathbf{ii} + \mathbf{jj}}{\sqrt{+2}},\ \mathbf{F}_2 = \frac{\mathbf{ii} - \mathbf{jj}}{\sqrt{-2}},\ \mathbf{F}_3 = \frac{\mathbf{ij} + \mathbf{ji}}{\sqrt{-2}} \tag{85}$$

and also write

$$x = \frac{\alpha_{11} + \alpha_{22}}{\sqrt{+2}},\ y = \frac{\alpha_{22} - \alpha_{11}}{\sqrt{-2}},\ z = \alpha_{12}\sqrt{-2} \tag{86}$$

when we shall have identically

$$\alpha_{11}\mathbf{ii} + \alpha_{12}\,(\mathbf{ij} + \mathbf{ji}) + \alpha_{22}\mathbf{jj} = x\mathbf{F}_1 + y\mathbf{F}_2 + z\mathbf{F}_3 = \varphi \tag{87}$$

The system $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ is now taken as a set of rectangular unit vectors in ordinary space. Thus any binary quadric corresponds to a vector drawn from a stipulated origin in space. As a single but important example of very numerous correspondences which arise, we have

$$\varphi^*\varphi = x^2 + y^2 + z^2 = 2\,(\alpha_{11}\alpha_{22} - \alpha_{12}^2) \tag{88}$$

This brief outline renders scant justice to the very beautiful system of Waelsch, from which it differs in slight details, owing to difference in definitions and point of view. It differs in particular by identification of the dyadic with the desired vector through the star product and the relation (87). The essential thing is the correspondence set up between geometry and invariant theory by relations of the type (88).

More generally, if we take the ordinary dyadics of Gibbs, making $K = 1$, $N = 3$, with unit vectors **i**, **j**, **k**, it is easy to verify that the system of nine dyadics given by

$$\mathbf{F}_1\sqrt{2} = \mathbf{ii}+\mathbf{jj},\ \mathbf{F}_2\sqrt{-4} = \mathbf{kk}-2\mathbf{ii},\ \mathbf{F}_3\sqrt{-4} = 2\mathbf{jj}-\mathbf{kk},\ \mathbf{F}_4\sqrt{-2} = \mathbf{ij}+\mathbf{ji}$$

$$\mathbf{F}_5\sqrt{-2} = \mathbf{jk}+\mathbf{kj},\ \mathbf{F}_6\sqrt{-2} = \mathbf{ki}+\mathbf{ik},\ \mathbf{F}_7\sqrt{2} = \mathbf{ij}-\mathbf{ji},\ \mathbf{F}_8\sqrt{2} = \mathbf{jk}-\mathbf{kj},\ \mathbf{F}_9\sqrt{2} = \mathbf{ki}-\mathbf{ik}$$

are normal and orthogonal under star product. As already noted, the star product in this special case of two dyadics φ and θ is the scalar $(\varphi\overset{\times}{\times}\theta)_S$ of Gibbs. If we wish to restrict ourselves to self-conjugate dyadics we have merely to drop $\mathbf{F}_7$, $\mathbf{F}_8$, and $\mathbf{F}_9$ from the list. It would be interesting to work out the analogy between the scalars called above $((\varphi^p))_S$, which might be symbolically written $((\mathbf{ab}))^p{}_S$, and the invariants of ternary quadrics written in the usual symbolic language.

We see however that systems orthogonal under star product will always be possible, because to every double polyad $\mathbf{E}_r\mathbf{E}_s$ corresponds always at least one other double polyad not orthogonal to the first.

13. Scalars formed by Star Multiplication.

In what follows I shall, for simplicity of language, take $K = 1$, so that double polyadics become dyadics, and polyadics become vectors. This imposes no formal limitation, and can be extended to the most general case with slight verbal change.

Supposing N to have any value, we assume $\mathbf{F}_1, \mathbf{F}_2, \cdots \mathbf{F}_\nu$ a set of linearly independent dyadics orthogonal under star product. Any other dyadics $\mathbf{A}$, $\mathbf{M}$, $\mathbf{B}$, or $\mathbf{N}$ can be expressed in terms of these. Let φ_1 and φ_2 as before be a pair of double dyadics $\Sigma\mathbf{AM}$ and $\Sigma\mathbf{BN}$. The star product of $\mathbf{AM}$ into $\mathbf{BN}$ is defined to be $\mathbf{A}(\mathbf{M}^*\mathbf{B})\mathbf{N}$ which is the same as $\mathbf{ANM}^*\mathbf{B}$. Thus $\varphi_1{}^*\varphi_2$ is a new double dyadic. If we regard $\varphi^*\varphi$ and $\varphi^*\varphi^*\varphi$ etc. as a new type of power abbreviated φ^{*2}, φ^{*3}, etc. no further proof is required to show that φ^* satisfies a Hamilton-Cayley equation similar in form to (28),

$$\varphi^{*\nu} - m_1{}^*\varphi^{*(\nu-1)} + \cdots + (-1)\, m^*{}_\nu I'' = 0 \tag{89}$$

nor to show that the star idemfactor I'' is identical with $\Sigma\mathbf{F}_i\mathbf{F}_i$. The whole of Art. 3 might be repeated at this point, replacing multiple dot product by star product and $\mathbf{E}$ by $\mathbf{F}$. The coefficients $m^*{}_p$ in (89) will be formed as in Art. 5, replacing multiple dot by star product. We shall also have a set of scalars

$$(\varphi_1, \varphi_2, \cdots, \varphi_p)_S{}^* \tag{90}$$

analogous with (36). For example

$$(\mathbf{AM}, \mathbf{BN})_S{}^* = \mathbf{A}^*\mathbf{MB}^*\mathbf{N} - \mathbf{A}^*\mathbf{NB}^*\mathbf{M} \tag{91}$$

which differs from (32) in having star in place of colon. It also differs in having no analogue when $\mathbf{A}$, $\mathbf{M}$, $\mathbf{B}$, $\mathbf{N}$ are mere vectors, whereas

(32) would still hold if single dot be written for colon. As before remarked, the star product is essentially double.

For further comparison, suppose $N = 2$, with the notation of Art. 10.

$$\begin{aligned}(\mathbf{AM}, \mathbf{BN})^*_S = {} & (a_{11}m_{22} - a_{12}m_{21} - a_{21}m_{12} + a_{22}m_{11})\,(b_{11}n_{22} - b_{12}n_{21} - b_{21}n_{12} \\ & + b_{22}n_{11}) \\ & - (a_{11}n_{22} - a_{12}n_{21} - a_{21}n_{12} + a_{22}n_{11})\,(b_{11}m_{22} - b_{12}m_{21} \\ & - b_{21}m_{12} + b_{22}m_{11}) \qquad (92)\end{aligned}$$

which, like (64), reduces to 24 scalar terms by expanding.

The method of cubic determinants might be applied to star scalars, but with less ease, because a system of dyadics orthogonal under star product is in general imaginary.

Identities like (44), (42), (45), (47), (48), and (54) to (61) inclusive may be proved in quite parallel steps, with appropriate change of notation. With exception of (54), (56), and (57), these identities do not imply the presence of $\sqrt{-1}$ even when we change to star product. While the theory of invariants lies outside the scope of this paper, it may be noted that star scalars, like those of (36), are by nature of their definition, invariant under transformation of the fundamental units, and some of them are invariant in a much wider sense.

14. Identities with the Star Idemfactor.

If we have a system of linearly independent dyadics $\mathbf{A}_1, \mathbf{A}_2, \cdots \mathbf{A}_\nu$ their reciprocals with respect to star product may be called $\mathbf{A''}_1$, $\mathbf{A''}_2, \cdots, \mathbf{A''}_\nu$. By steps as in Art. (3) we may show that $\Sigma\mathbf{AA''}$ is identically the idemfactor $\Sigma\mathbf{FF}$. It is evident by Art. 12 that the star idemfactor is real. Taking $N = 3$ with unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, it is easy to verify that the star idemfactor

$$I'' = \Sigma\left[\frac{1}{2}\mathbf{ii}(\mathbf{jj} + \mathbf{kk} - \mathbf{ii}) - (\mathbf{ijji} + \mathbf{jiij})\right] \qquad (93)$$

because $I''^*\,\mathbf{ii} = \mathbf{ii}$, $I''^*\,\mathbf{ij} = \mathbf{ij}$, etc. In fact since, as operators,

$$I''^* = I: \qquad (94)$$

both being identical operations, we must have

$$\mathbf{A''}_\nu{}^* = \mathbf{A'}_\nu: \qquad (95)$$

where $\mathbf{A''}_\nu$ and $\mathbf{A'}_\nu$ are respectively the star and dot reciprocals of $\mathbf{A}_\nu$ in the set of linearly independent dyadics $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_\nu$. Operating with both sides of (95) on the dot idemfactor gives

$$\mathbf{A''}_r{}^* I = \mathbf{A'}_r \tag{96}$$

or in words: the dot reciprocal is the star product of star reciprocal and dot idemfactor. Similarly by operating on the star idemfactor,

$$\mathbf{A''}_r = \mathbf{A'}_r : I'' \tag{97}$$

or: the star reciprocal is the dot product of dot reciprocal and star idemfactor. Thus the star reciprocal of $\mathbf{ii}$ is $\mathbf{ii}: I''$ or $\frac{1}{2}(\mathbf{jj} + \mathbf{kk} - \mathbf{ii})$, and the star reciprocal of $\mathbf{ij}$ is $-\mathbf{ji}$ etc.

Star multiplication may always be replaced by dot multiplication by introducing the double dyadic I^*I between the factors. For if $\mathbf{M}$ and $\mathbf{N}$ are any two dyadics we have $\mathbf{M}: I = \mathbf{M}$ and $I: \mathbf{N} = \mathbf{N}$, hence

$$\mathbf{M}^*\mathbf{N} = \mathbf{M} : I^*I : \mathbf{N} \tag{98}$$

We also observe that I^*I is the reciprocal (double dyadic) of I'' with respect to double dot multiplication, that is

$$I'' : I^*I = I''^*I = I \tag{99}$$

In a similar manner double dot multiplication can be changed to star by the operation $I'' : I''$ for

$$\mathbf{M} : \mathbf{N} = \mathbf{M}^* I'' : I''^* \mathbf{N} \tag{100}$$

and $I'' : I''$ is the reciprocal of I with respect to star multiplication,

$$I^*I'' : I'' = I : I'' = I'' \tag{101}$$

It is thus plain that, in spite of the imaginary character of star orthogonality, the reciprocals of a set of real dyadics are real, and the reciprocal, (or inverse), of a real double dyadic, if it exist, is also real.

15. Illustration of Various Types of Multiplication.

We may close this discussion of the general properties of double dyadics by a few examples of the types of product considered. Consider the two tetrads $\mathbf{ijji}$ and $\mathbf{jiij}$. If they be taken as polyadics of order 4 and multiplied by multiple dot product we have

$$(\mathbf{ijji}) : (\mathbf{jiij}) = 0 \tag{102}$$

where colon means multiple dot product.

If they be taken as a pair of double dyadics the scalar (36) is

$$((\mathbf{ij} \mid \mathbf{ji}, \mathbf{ji} \mid \mathbf{ij}))_S = \mathbf{ij} : \mathbf{jiji} : \mathbf{ij} - \mathbf{ij} : \mathbf{ijji} : \mathbf{ji} = -1 \tag{103}$$

where the colon stands for double dot product, for we assume the case $K = 2$, hence might also have written $(\mathbf{ij} \mid \mathbf{ji})^*(\mathbf{ji} \mid \mathbf{ij})$ for this scalar.

If we assume $K = 1$ we return to the case of Art. 13 and 14 and have

$$(\mathbf{ij} \mid \mathbf{ji}, \mathbf{ji} \mid \mathbf{ij})^*_S = \mathbf{ij}^*\mathbf{jiji}^*\mathbf{ij} - \mathbf{ij}^*\mathbf{ijji}^*\mathbf{ji} = +1 \tag{104}$$

and finally if we had merely four dyads **ij**, **ji**, **ji**, and **ij**, the scalar

$$((\mathbf{ij}, \mathbf{ji}, \mathbf{ji}, \mathbf{ij}))_S = 0 \tag{105}$$

since all such scalars vanish unless the antecedents are distinct.

Again, we may form double and mixed products of double dyadics. Assuming $K = 1$ the star product written singly will refer to the dyadic factors. A double star will refer to antecedents and consequents as in the usual double product, thus if **A**, **M**, **B**, **N** are dyadics

$$\mathbf{AM}^{**}\mathbf{BN} = \mathbf{A}^*\mathbf{BM}^*\mathbf{N} \tag{106}$$

and we may have

$$\mathbf{AM}^*\mathbf{:BN} = \mathbf{A}^*\mathbf{BM:N} \tag{107}$$

and

$$\mathbf{AM:}^*\mathbf{BN} = \mathbf{A:BM}^*\mathbf{N} \tag{108}$$

while

$$\mathbf{AM::BN} = \mathbf{A:BM:N} \tag{109}$$

It is evident that, regarding the double dyadic as an element, we could start a fresh investigation of each of these types of product with respect to orthogonality, idemfactors, and so on ad inf. But we have now ample tools for the applications it is desired to make.

PART II. THE SOLUTION OF THE LINEAR MATRIX EQUATION.

16. Transformation from Matrices to Double Dyadics.

Any matrix A of order N is defined as a binary assemblage of elements a_{ik} where subscripts run from 1 to N. We add two matrices by adding corresponding elements. We multiply by the rule

$$(AB)_{ik} = \Sigma a_{is} b_{sk} \ (s = 1, 2, \cdots, N) \tag{110}$$

If $A_1, A_2, \cdots, A_h$ and $B_1, B_2 \cdots, B_h$ and C are known matrices while x is a required matrix, the linear matrix equation may be written

$$\theta(x) = A_1 x B_1 + A_2 x B_2 + \cdots + A_h x B_h = C. \tag{111}$$

The number of terms h on the left is said to denote the *extent* of the operation θ or of the equation.

It is known that any matrix A is equivalent to a dyadic $\mathbf{A}$ or $\Sigma\mathbf{aa}'$ where $\mathbf{a}$ and $\mathbf{a}'$ are vectors. Any term $A\,x\,B$ is therefore equivalent to a sum of terms of the form $\mathbf{aa}' \cdot \mathbf{X} \cdot \mathbf{bb}'$ where $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, and $\mathbf{b}'$ are vectors in space of N dimensions and $\mathbf{X}$ is a required dyadic. Since by the definition of double dot product we have $\mathbf{ab:xy} = \mathbf{a}\cdot\mathbf{xb}\cdot\mathbf{y}$, and since $\mathbf{X}$ might be written $\Sigma\mathbf{xy}$ we shall have

$$\mathbf{aa}' \,\mathbf{X}\, \mathbf{bb}' = \mathbf{ab}'\mathbf{a}'\mathbf{b}:\mathbf{X} \tag{112}$$

which is the transformation at the basis of the method to be exhibited. Thus the linear matrix equation is equivalent to

$$\varphi:\mathbf{X} = \Sigma\mathbf{MN}:\mathbf{X} = C \tag{113}$$

where φ is a dyadic whose antecedents and consequents are themselves dyadics, that is a double dyadic.

Since $\varphi:\mathbf{X} = \theta(x)$, θ and φ satisfy the same Hamilton-Cayley equation. One method of solving the linear matrix equation is therefore by putting

$$x = \theta^{-1}C = \frac{1}{m_n}[\theta^{n-1} - m_1\theta^{n-2} + \cdots + (-1)^{n-1}m_{n-1}I](-1)^{n-1}C \tag{114}$$

where $m_1 \cdots m_n$ are the same coefficients as in (28) and where m_n is assumed not zero.

17. Transformation back to Matrices.

If the matrices A and B are known in such form that transformation to (113) is convenient, any of the methods exhibited in Part I may be used to calculate the coefficients m. A general solution, however, demands ability to compute them directly from $A_1 \cdots A_h$ and $B_1 \cdots B_h$ without the necessity of first forming φ by the rule (112). In other words we need to transform the scalars (36) from the language of double dyadics to that of matrices, taking account of (112). This may be conveniently accomplished by a partly symbolic notation, omitting subscripts and summation signs which refer to the terms $1, 2, \cdots h$ of the matrix equation but occasionally preserving those which refer to the $p!$ terms of the expansion of (35). We thus write symbolically

$$\varphi = \mathbf{ab}' \mid \mathbf{a}'\mathbf{b} = \mathbf{MN} \tag{115}$$

and
$$\begin{aligned} m_1 &= \mathbf{ab'}:\mathbf{a'b} \\ &= \mathbf{a}\cdot\mathbf{a'b}\cdot\mathbf{b'} \\ &= A_S B_S \end{aligned} \tag{116}$$

where A_S and B_S are respectively the first coefficients in the Hamilton-Cayley equations for A and B; or, what is the same thing, they are the sums of elements along the main diagonal of these matrices. Stated in complete form the result is

$$m_1 = A_{1S}B_{1S} + A_{2S}B_{2S} + \cdots + A_{hS}B_{hS} \tag{117}$$

As remarked earlier, the symbolic process is valid by virtue of the distributive character of all the steps involved. Thus for m_2 we may put, in abbreviated language,

$$\begin{aligned} 2m_2 &= ((\varphi, \varphi))_S = ((\mathbf{ab'} \mid \mathbf{a'b}, \mathbf{cd'} \mid \mathbf{c'd}))_S \\ &= \mathbf{ab'}:\mathbf{a'bcd'}:\mathbf{c'd} - \mathbf{ab'}:\mathbf{c'dcd'}:\mathbf{a'b} \\ &= \mathbf{a}\cdot\mathbf{a'b}\cdot\mathbf{b'c}\cdot\mathbf{c'd}\cdot\mathbf{d'} - \mathbf{a}\cdot\mathbf{c'b'}\cdot\mathbf{dc}\cdot\mathbf{a'd'}\cdot\mathbf{b} \end{aligned} \tag{118}$$

Now the first term on the right is the product of the scalars of the four dyadics or matrices which enter; that is, it is symbolically $A_S B_S C_S D_S$ or $A_{iS}B_{iS}A_{jS}B_{jS}$. The second term may be further transformed

$$\begin{aligned} \mathbf{a}\cdot\mathbf{c'b'}\cdot\mathbf{dc}\cdot\mathbf{a'd'}\cdot\mathbf{b} &= [\mathbf{a}\cdot\mathbf{c'a'}\cdot\mathbf{c}]\,[\mathbf{b}\cdot\mathbf{d'b'}\cdot\mathbf{d}] \\ &= [\mathbf{aa'}\cdot\mathbf{cc'}]_S\,[\mathbf{bb'}\cdot\mathbf{dd'}]_S \\ &= [AC]_S\,[BD]_S \\ &= [A_iA_j]_S\,[B_iB_j]_S \end{aligned} \tag{119}$$

where $[A_iA_j]_S$ is the first coefficient in the Hamilton-Cayley equation for the matrix A_iA_j, that is, for the product of A_i and A_j, it is the sum of elements on the main diagonal of this product; and similarly for $[B_iB_j]_S$. Stated in complete form the result is

$$2m_2 = \Sigma[A_{iS}B_{iS}A_{jS}B_{jS} - (A_iA_j)_S(B_iB_j)_S],\ [i, j = 1, 2, \cdots, h] \tag{120}$$

the summation being now with respect to all possible pairs of terms selected from the left side of the original equation (111). It is to be particularly noted that the summation includes the cases $i = j$; in other words, by "pairs of terms" is meant not merely different terms but also the same term taken twice. The reason is not far to seek. By the transformation (112) any term AxB was changed into a sum of terms of the form $\mathbf{MN}:\mathbf{X}$, but not, in general, into a single such term. It follows that, although $((\mathbf{MN}, \mathbf{MN}))_S$, which is the star product of a single term $\mathbf{MN}$ into itself, must vanish, yet a scalar $A^2_S B^2_S - (A^2)_S(B^2)_S$ *symbolically derived from it* does not in general vanish.

This fact will be fully exemplified below. A symbolic equation implies, not the equality of individual terms each to each, but equality of the sums of all possible terms of the equated forms.

Following a similar procedure to find m_3 we have

$$\begin{aligned} 6m_3 = ((\varphi, \varphi, \varphi))_S = ((\mathbf{ab'} \mid \mathbf{a'b}, \mathbf{cd'} \mid \mathbf{c'd}, \mathbf{ef'} \mid \mathbf{e'f}))_S, \text{ symbolically,} \\ = \mathbf{a \cdot a'b \cdot b'c \cdot c'd \cdot d'e \cdot e'f \cdot f'} - \mathbf{a \cdot a'b \cdot b'[c \cdot e'c' \cdot e]\,[d' \cdot fd \cdot f']} \\ - \mathbf{c \cdot c'd \cdot d'[e \cdot a'e' \cdot a]\,[f' \cdot bf \cdot b']} - \mathbf{e \cdot e'f \cdot f'[a \cdot c'a' \cdot c]\,[b' \cdot db \cdot d']} \\ + \mathbf{(ab' : c'd)\,(cd' : e'f)\,(ef' : a'b)} + \mathbf{(ab' : e'f)\,(cd' : a'b)\,(ef' : c'd)} \end{aligned} \tag{121}$$

by direct application of the definition of Art. 5, and by grouping the dot products in the second, third, and fourth terms after the manner of (119). The first or leading term is evidently the same symbolically as $A_{iS}B_{iS}A_{jS}B_{jS}A_{kS}B_{kS}$. The next three terms taken together are symbolically $-\Sigma A_{iS}B_{iS}(A_jA_k)_S(B_jB_k)_S$. In transforming the fifth term the procedure, as in all cases, will be to group into one factor those vectors which correspond to prefactors A in the original equation, and into another factor those which correspond to the post-factors B. Thus

$$\begin{aligned} \mathbf{(ab' : c'd)\,(cd' : e'f)\,(ef' : a'b)} &= \mathbf{(a \cdot c'c \cdot e'e \cdot a')\,(b' \cdot dd' \cdot ff' \cdot b)} \\ &= \mathbf{(aa' \cdot ee' \cdot cc')}_S\mathbf{(bb' \cdot dd' \cdot ff')}_S \\ &= (A_iA_kA_j)_S(B_iB_jB_k)_S \end{aligned} \tag{122}$$

It is especially worthy of remark that the order of the matrices in one factor is the reverse of that in the other, a consequence of the transformation (112). When there are only two matrices in each factor, as in (119), it was of no importance whether we wrote $(AC)_S$ or $(CA)_S$, for these are equal: the scalar of the product of two matrices is independent of their order, because $\mathbf{(aa' \cdot cc')}_S = \mathbf{a \cdot c'a' \cdot c} = \mathbf{(cc' \cdot aa')}$. But by similar reasoning we see that the scalar of the product of several matrices depends on their cyclic order, as is well known.

In the same way the last term of (121) becomes $(A_iA_jA_k)_S(B_iB_kB_j)_S$. Collecting results, the complete statement for m_3 is

$$\begin{aligned} 6m_3 = A_{iS}A_{jS}A_{kS}B_{iS}B_{jS}B_{kS} - A_{iS}B_{iS}(A_jA_k)_S(B_jB_k)_S - \\ A_{jS}B_{jS}(A_kA_i)_S(B_kB_i)_S \\ - A_{kS}B_{kS}(A_iA_j)_S(B_iB_j)_S + (A_iA_kA_j)_S(B_iB_jB_k)_S + \\ (A_iA_jA_k)_S(B_iB_kB_j)_S \end{aligned} \tag{123}$$

which is to be summed over all possible sets of three terms from the h terms of the original equation (111), allowing repetitions for the reason

already noted under m_2; that is $i, j, k = 1, 2, \cdots, h$. Such a summation will hereafter be referred to as *summation over the extent of the equation,*[12] and is implied in every symbolic equation.

In general we may express m_p symbolically by the equation

$$p!m_p = ((\mathbf{a}_1\mathbf{b}'_1 \mid \mathbf{a}'_1\mathbf{b}_1, \mathbf{a}_2\mathbf{b}'_2 \mid \mathbf{a}'_2\mathbf{b}_2, \cdots, \mathbf{a}_p\mathbf{b}'_p \mid \mathbf{a}'_p\mathbf{b}_p))_S \qquad (124)$$

Expanding by the definition of Art. 5, the leading term is the product of p factors of the form $\mathbf{a}_r\mathbf{b}'_r : \mathbf{a}'_r\mathbf{b}_r$, because the leading term is made without interchanges among the consequents. These factors are the same as $\mathbf{a}_r \cdot \mathbf{a}'_r\mathbf{b}_r \cdot \mathbf{b}'_r$ and symbolically the same as $A_{rS}B_{rS}$. Thus if we have a set of p subscripts $i, j, k, \cdots, r, s, t$, the leading term in the development of $p!m_p$ is

$$A_{iS}B_{iS}A_{jS}B_{jS}A_{kS}B_{kS}\cdots A_{rS}B_{rS}A_{sS}B_{sS}A_{tS}B_{tS} \qquad (125)$$

which is to be summed over the extent of the equation (11).

The other terms in the expansion of (124) are obtained by making all possible interchanges among the consequents, according to the definition of Art. 5. Therefore so long as we maintain the polyadic notation any term is a product of factors of the form $\mathbf{a}_r\mathbf{b}'_r : \mathbf{a}'_s\mathbf{b}_s$, which is the same as $\mathbf{a}_r \cdot \mathbf{a}'_s\mathbf{b}'_r \cdot \mathbf{b}_s$. The vectors $\mathbf{a}_r$ and $\mathbf{a}'_s$ might correspond to different matrices, likewise $\mathbf{b}'_r$ and $\mathbf{b}_s$. Hence such a factor cannot in general be translated into matrix notation if considered by itself. For it is of the essence of this transformation that every vector $\mathbf{a}_r$ be associated with its mate $\mathbf{a}'_r$ and likewise every $\mathbf{b}_r$ with $\mathbf{b}'_r$; then $\mathbf{a}_r\mathbf{a}'_r$ is symbolically equivalent to A_r and $\mathbf{b}_r\mathbf{b}'_r$ to B_r. If, however, by a simple interchange of a pair of consequents in the polyadic expression, we obtain a pair of factors $(\mathbf{a}_r\mathbf{b}'_r : \mathbf{a}'_s\mathbf{b}_s)$ $(\mathbf{a}_s\mathbf{b}'_s : \mathbf{a}'_r\mathbf{b}_r)$, these may be developed as in (118) and (119) and yield the two factors $(A_rA_s)_S(B_rB_s)_S$. Such factors occurred already in the second term of m_2 and in the second, third, and fourth terms of m_3.

If three consequents change places among themselves, we obtain a product of three factors of the form $(\mathbf{a}_r\mathbf{b}'_r : \mathbf{a}'_s\mathbf{b}_s)$ $(\mathbf{a}_s\mathbf{b}'_s : \mathbf{a}'_t\mathbf{b}_t)$ $(\mathbf{a}_t\mathbf{b}'_t : \mathbf{a}'_r\mathbf{b}_r)$. It is important to notice that *the order of subscripts in the vectors* $\mathbf{a}$ *is the same as that in the vectors* $\mathbf{b}$ *but the order of accents is reversed.* It follows that when we develop after the manner of (122) the vectors $\mathbf{a}$ yield the product of dot products $\mathbf{a}_r \cdot \mathbf{a}'_s\mathbf{a}_s \cdot \mathbf{a}'_t\mathbf{a}_t, \mathbf{a}'_r$, while the vectors $\mathbf{b}$ yield the product $\mathbf{b}'_r \cdot \mathbf{b}_s\mathbf{b}'_s \cdot \mathbf{b}_t\mathbf{b}'_t, \mathbf{b}_r$, *where the order of subscripts is the same, but the accents are on the first vector in each dot product, instead of the second.* These factors are equivalent in matrix form to $(A_rA_tA_s)_S$ and $(B_rB_sB_t)_S$, where the *order of sub-*

scripts in the postfactors B is the same as in the polyadic term, but the order of subscripts in the prefactors A is the reverse of the order in the postfactors B. For in the polyadic expression $\mathbf{b}'_s$ follows its mate $\mathbf{b}_s$, and $\mathbf{b}'_t$ follows $\mathbf{b}_t$, and if we rewrite the factor in the form $(\mathbf{b}_r\mathbf{b}'_r \cdot \mathbf{b}_s\mathbf{b}'_s \cdot \mathbf{b}_t\mathbf{b}'_t)_S$ then $\mathbf{b}'_r$ also follows its mate $\mathbf{b}_r$, whence directly the matrix factor $(B_rB_sB_t)_S$. But in the polyadic expression as it stands, each $\mathbf{a}'$, when we follow the same steps, *precedes* its mate on account of the reversal of the order of accents. Thence follows the reversal of order of matrices in the corresponding factor. Transformations of this type already occurred in the last two terms of m_3.

All the italicized statements in the above discussion are true no matter how many consequents have changed places among themselves. We may therefore write a general rule for the formation of m_p.

Rule for forming $p!m_p$ where m_p is the coefficient of θ^p in the Hamilton-Cayley equation for θ.

Let there be p subscripts $i, j, k, \cdots, r, s, t$, each of which may have any value from 1 to h. Choosing a particular set of values for these subscripts, we form a group of $p!$ terms as follows: the leading term is $A_{iS}B_{iS}A_{jS}B_{jS}A_{kS}B_{kS}\cdots A_{rS}B_{rS}A_{sS}B_{sS}A_{tS}B_{tS}$. The other terms are formed from the leading term by first interchanging the postfactors B in all ways, while the prefactors A are at first left fixed in position. If a particular postfactor B is left in position, it yields in the corresponding term a factor B_S and its prefactor yields A_S *precisely as in the leading term.* If a pair of postfactors as B_r and B_q change places, there results in the corresponding term a factor $(A_rA_q)_S$ $(B_qB_r)_S$. If three postfactors, as B_k, B_r, and B_s, change places so that their new order is B_r, B_s, B_k, there results in the corresponding term a factor $(A_kA_sA_r)_S(B_rB_sB_k)_S$ *where the order of prefactors is the reverse of the final order of the postfactors.* In general if any group of postfactors change places among themselves so that their final order is

$$B_kB_r\cdots B_sB_j$$

there results in the corresponding term a factor $(A_jA_s\cdots A_rA_k)_S$ $(B_kB_r\cdots B_sB_j)_S$ where the order of prefactors is the reverse of the final order of postfactors.

The group of $p!$ terms thus obtained is to be summed over the extent of the linear matrix equation.

18. Character of the Coefficients as Algebraic Polynomials.

It is evident from the form of the leading term that in every term of the expansion of m_p will be found p prefactors and p postfactors,

whether alike or different. Hence m_p is homogeneous and of degree $2p$ in the scalar elements of the given matrices which define θ.

Furthermore m_p is non-homogeneous and of degree p (in general) in the scalar elements of any particular matrix. For if the p subscripts which occur in a group of terms are all alike, the p prefactors and the p postfactors are all alike.

Again, each group of terms, corresponding to a particular choice of p subscripts, is homogeneous in the scalar elements of each of the matrices which occur in the group. The degree in any one matrix is determined by the number of times its subscript has been chosen in making up p subscripts. The degree in A_r is the same as that in B_r.

We thus see that each group of terms formed by the above rule is made up of terms essentially *unlike* the terms of all the other groups, each group, in other words, constitutes a distinct polynomial in the scalar elements of certain of the given matrices. Within the group, however, various terms may be like one another, and simplifications may occur, particularly when special values are assigned to some of the matrices; for example, when some of them are allowed to become the idemfactor.

19. The Equation of Extent Unity and Order Two.

As an example, closely allied to that of Art. 10, let us take all given matrices to be of the second order, and, to begin with, let $h = 1$, so that the linear matrix equation reduces to the simple form

$$AxB = C \tag{126}$$

and can be at once solved as $x = A^{-1}CB^{-1}$. Forming the m_p by our rule we have $m_1 = A_S B_S$ which is $(a_{11} + a_{22})\,(b_{11} + b_{22})$. Next $2m_2 = A^2{}_S B^2{}_S - (A^2)_S (B^2)_S$.

The scalar $(A^2)_S$ may be expressed in terms of simpler quantities by using the Hamilton-Cayley equation for the matrix A. Let the coefficients in this equation be A_S and A''_S so that

$$A^2 = A_S A - A''_S I \tag{127}$$

where I is the identical matrix of the second order. Taking the scalar of both sides of the equation we have

$$(A^2)_S = A^2{}_S - 2A''_S \tag{128}$$

and a similar equation for $(B^2)_S$. Substituting in the expression for $2m_2$ we find on simplifying

$$m_2 = A^2{}_S B''_S - 2A''_S B''_S + A''_S B^2{}_S \qquad (129)$$

Since the matrices are of the second order, A''_S and B''_S are the determinants of their respective matrices; hence on substituting,

$$\begin{aligned} m_2 &= (a_{11} + a_{22})^2 (b_{11}b_{22} - b_{12}b_{21}) - 2(a_{11}a_{22} - a_{12}a_{21}) (b_{11}b_{22} - b_{12}b_{21}) \\ &\quad + (a_{11}a_{22} - a_{12}a_{21}) (b_{11} + b_{22})^2 \\ &= (a^2{}_{11} + a^2{}_{22}) (b_{11}b_{22} - b_{12}b_{21}) + (b^2{}_{11} + b^2{}_{22}) (a_{11}a_{22} - a_{12}a_{21}) \\ &\quad + 2a_{11}a_{22}b_{11}b_{22} - 2a_{12}a_{21}b_{12}b_{21} \end{aligned} \qquad (130)$$

Again

$$6m_3 = A^3{}_S B^3{}_S - 3A_S B_S (A^2)_S (B^2)_S + 2(A^3)_S (B^3)_S \qquad (131)$$

by the rule.* But $A^3 = A_S A^2 - A''_S A$ hence

$$(A^3)_S = A^3{}_S - 3A_S A''_S \qquad (132)$$

by (128), and similarly for B. Substituting values in (131) from (128) and (132) to get rid of scalars of powers of matrices we find

$$\begin{aligned} m_3 &= A_S B_S A''_S B''_S \qquad (133) \\ &= (a_{11} + a_{22}) (b_{11} + b_{22}) (a_{11}a_{22} - a_{12}a_{21}) (b_{11}b_{22} - b_{12}b_{21}) \end{aligned}$$

and finally $24m_4 = A^4{}_S B^4{}_S - 6A^2{}_S B^2{}_S (A^2)_S (B^2)_S + 3(A^2)^2{}_S (B^2)^2{}_S$

$$+ 8A_S B_S (A^3)_S (B^3)_S - 6(A^4)_S (B^4)_S \qquad (134)$$

But $A^4 = A_S A^3 - A''_S A^2$ hence

$$(A^4)_S = A^4{}_S - 4A^2{}_S A''_S + 2A''{}^2_S \qquad (135)$$

so that by eliminating the scalars of powers from (134)

$$m_4 = A''{}^2_S B''{}^2_S \qquad (140)$$

Now in the present example $\theta^2 C = A^2 C B^2$, $\theta^3 C = A^3 C B^3$, etc. Hence

$$\begin{aligned} x = \theta^{-1} C = \frac{1}{A''{}^2_S B''{}^2_S} [&A_S B_S A''_S B''_S C - (A^2{}_S B''_S - 2A''_S B''_S \\ &+ A''_S B^2{}_S) ACB + A_S B_S A^2 C B^2 - A^3 C B^3] \end{aligned} \qquad (141)$$

Simplifying by the aid of the Hamilton-Cayley equations for A and B,

$$\begin{aligned} x &= \frac{1}{A''_S B''_S} [A_S B_S C - B_S AC - A_S BC + ACB] \\ &= \frac{1}{A''_S B''_S} [A_S I - A] C [B_S I - B] = A^{-1} C B^{-1}, \end{aligned}$$

checking with the known solution.

20. The Equation of Extent Two and Order Two.

The following considerations will suggest how the work of solution may be arranged when the equation is of higher extent. We have seen that any coefficient m_p is a sum of groups of terms. Each group is a homogeneous polynomial. Suppose a choice of subscripts in which i occurs a times, j occurs b times, etc. Let the development of terms, under the rule, corresponding to this choice of subscripts, be denoted by $G(i^a j^b \cdots)$. Each choice of subscripts will occur a number of times equal to the coefficient of the corresponding term in the expansion of

$$(i + j + k + \cdots + r + s + t)^p. \tag{142}$$

Thus the entire development may be systematically carried out.

For example, take

$$A_i x B_i + A_j x B_j = C \tag{143}$$

and form m_4. With the above notation we shall have

$$m_4 = G(i^4) + 4G(i^3 j) + 6G(i^2 j^2) + 4G(ij^3) + G(j^4) \tag{144}$$

By the preceding example we have $G(i^4) = A''_i{}^2{}_S B''_i{}^2{}_S$ and similarly for $G(j^4)$. By the rule we have

$$\begin{aligned} G(i^3 j) = {} & A^3{}_{iS} A_{jS} B^3{}_{iS} B_{jS} - 3A^2{}_{iS}(A_i A_j)_S B^2{}_{iS}(B_i B_j)_S \\ & - 3A_{iS} A_{jS}(A^2{}_i)_S B_{iS} B_{jS}(B^2{}_i) + 3(A^2{}_i)_S(A_i A_j)_S(B^2{}_i)_S(B_i B_j)_S \\ & + 6A_{iS}(A^2{}_i A_j)_S B_{iS}(B^2{}_i B_j)_S + 2A_{jS}(A^3{}_i)_S B_{jS}(B^3{}_i)_S \\ & - 6\,(A^3{}_i A_j)_S(B^3{}_i B_j)_S \end{aligned} \tag{145}$$

Brevity will be gained in notation, while nothing is lost in explicitness, if, in such expressions, we indicate only the matrices A which are prefactors in the original equation, remembering that the order of subscripts among postfactors, when more than two matrices are multiplied, is the reverse of that for prefactors. With this understanding we may also omit the letter A and the subscript S. Thus (145) may be abbreviated

$$G(i^3 j) = i^3 j - 3i^2(ij) - 3ij(i^2) + 3(i^2)\,(ij) + 6i(i^2 j) + 2j(i^3) - 6(i^3 j) \tag{145a}$$

As a check, if A_j and B_j are replaced by the idemfactor of the second order, $G(i^3 j)$ should reduce to $6A_S B_S A''_S B''_S$ because m_4 becomes the determinant of $A(\ )B + (\ )$, and the terms of the third degree in A

must be the coefficient m_3 for the example of the last article. In fact we then have, remembering $I_S = 2$,

$$G(i^3j) = 4A^3{}_S B^3{}_S - 3A^3{}_S B^3{}_S - 12A_S(A^2)_S B_S(B^2)_S + 3A_S(A^2)_S B_S(B^2)_S + 6A_S(A^2)_S B_S(B^2)_S + 8(A^3)_S(B^3)_S - 6(A^3)_S(B^3)_S$$

which is the same as (131).

Developments like (145) are the same in form no matter what the order of the matrices involved. They may in general be simplified by the use of the Hamilton-Cayley equations for A and B.

$G(ij^3)$ may be obtained from $G(i^2j)$ by interchange of subscripts. For the middle term of (144) we have

$$G(i^2j^2) = i^2j^2 - i^2(j^2) - j^2(i^2) - 4ij(ij) + (i^2)(j^2) + 2(ij)^2 + 4i(ij^2) + 4j(i^2j) - 2(ijij) - 4(i^2j^2) \tag{146}$$

which by letting A_j and B_j be I of the second order should become $2A^2{}_S B^2{}_S - 2(A^2)_S(B^2)_S$ as is easily verified.

In simplifying, use is to be made of the identity

$$(A_i A_j)_S = A_{iS} A_{jS} - A^*{}_i A_j \tag{147}$$

where the last term is the star product of A_i and A_j as already defined. Since this scalar is an invariant of the two matrices it may well be abbreviated $A^*{}_{ij}$. Collecting and reducing results we find

$$m_4 = A''^2{}_{iS} B''^2{}_{iS} + A''{}_{iS} B''{}_{iS} A^*{}_{ij} B^*{}_{ij} + A''{}_{iS} A''{}_{jS} B^{*2}{}_{ij} + B''{}_{iS} B''{}_{jS} A^{*2}{}_{ij} - 2A''{}_{iS} A''{}_{jS} B''{}_{iS} B''{}_{jS} + A''{}_{jS} B''{}_{jS} A^*{}_{ij} B^*{}_{ij} + A''^2{}_{jS} B''^2{}_{jS} \tag{148}$$

and by similar processes,

$$m_3 = A_{iS} A''{}_{iS} B_{iS} B''{}_{iS} + A_{iS} B_{jS} B''{}_{iS} A^*{}_{ij} + A_{jS} B_{iS} A''{}_{iS} B^*{}_{ij} - A_{jS} B_{jS} A''{}_{iS} B''{}_{iS} + A_{jS} B_{jS} A''{}_{jS} B''{}_{jS} + A_{jS} B_{iS} B''{}_{jS} A^*{}_{iS} + A_{iS} B_{jS} A''{}_{jS} B^*{}_{ij} - A_{iS} B_{iS} A''{}_{jS} B''{}_{jS} \tag{149}$$

$$m_2 = A^2{}_{iS} B''{}_{iS} - 2A''{}_{iS} B''{}_{iS} + A''{}_{iS} B^2{}_{iS} + A_{iS} A_{jS} B^*{}_{ij} - A^*{}_{ij} B^*{}_{ij} + B_{iS} B_{jS} A^*{}_{ij} + A^2{}_{jS} B''{}_{jS} - 2A''{}_{jS} B''{}_{jS} + A''{}_{jS} B^2{}_{jS} \tag{150}$$

$$m_1 = A_{iS} B_{iS} + A_{jS} B_{jS} \tag{151}$$

whence (143) is completely solved.

[1] This symbolic method is due to H. B. Phillips: *Some invariants and covariants of ternary collineations,*" American Journal of Mathematics, **36**, 1914.

[2] Hamilton, *Lectures on Quaternions*, 1853, *Cayley, A Memoir on the theory of matrices*, 1858.

[3] E. B. Wilson, *On the theory of double products and strains in hyperspace*, Conn. Acad. Trans. **14**, 1908.

[4] C. L. E. Moore and H. B. Phillips, *The dyadics which occur in a point-space of three dimensions*, Proc. Amer. Acad. of Arts and Sci. **53**, 1918.

[5] For the elementary theory, Bocher's Algebra may be consulted.

[6] See note 3.

[7] For the laws of *p-way determinants* see Amer. Journal of Math. **40**, 1918, by Lepine Hall Rice.

[8] Compare Joly's Appendix to Hamilton's *Elements of Quaternions*.

[9] This result is an extension of Hamilton's invariant property of his coefficients.

[10] *Elements of Quaternions*, 2nd Ed., Vol. I, Art. 348.

[11] In a series of papers over many years. They are all, I think, listed in the bulletins of the Quaternion Association. See, in particular, Wien. Ber. **112**, 1903, pp. 645, 1091, and 1533.

[12] The term "extent" is due to Sylvester. I have elsewhere given a sketch of the present method in its relation to the work of Sylvester. Proc. Nat. Acad. of Sci. **8**, April, 1922.